Commutative Ring Without Unique Prime Factorizations

A commutative ring is a set R with addition, subtraction, and multiplication (with their usual properties)

Example: $\mathbb{Z}$

Consider $R = \{a + b\sqrt{5}i : a, b \in \mathbb{Z}\}$ $(i = \sqrt{-1})$

Divisibility: $x \mid y$ if $\exists q \in R\ (y = qx)$

Irreducible elements: $x \in R$ is irreducible if $\forall y, z \in R\ (x = yz \Rightarrow y = \pm 1 \text{ or } z = \pm 1)$

(or more generally, y or z has a multiplicative inverse in R)

Example: $1+\sqrt{5}i$, $1-\sqrt{5}i$, 2, and 3 are all irreducible.

29 is not irreducible as $29 = (3+2\sqrt{5}i)(3-2\sqrt{5}i)$

Factorizations: Every $x \in R \setminus \{0, -1, 1\}$ can be written as a product of irreducible elements but these factorizations may not be unique.

Example: $6 = 2 \cdot 3 = (1+\sqrt{5}i) \cdot (1-\sqrt{5}i)$