Proof of the Fundamental Theorem of Arithmetic

Fundamental Theorem of Arithmetic:
Let $p_1, p_2, p_3, \ldots$ be the primes in ascending order.
For all natural numbers n, there is a unique sequence $c_1, c_2, c_3, \ldots$ such that:

1) $\forall j \in \mathbb{N} \, (c_j \in \mathbb{N} \cup \{0\})$
2) $n = \prod_{j=1}^{\infty} p_j^{c_j}$
3) Only finitely many c_j are nonzero.

Key facts:

Lemma: $\forall n \in \mathbb{N} \setminus \{1\} \; \exists p \in \mathbb{P} \, (p \mid n)$ (i.e. every natural number greater than 1 is divisible by some prime number)

Lemma (prime property): $\forall p \in \mathbb{P} \; \forall x, y \in \mathbb{Z} \, (p \mid xy \rightarrow p \mid x \vee p \mid y)$

Proof of the Fundamental Theorem of Arithmetic

We'll prove the Fundamental Theorem of Arithmetic by induction.

Base case: $n=1$. $c_1=0, c_2=0, c_3=0, \ldots$ is the unique sequence of non-negative integers such that $1=\prod_{j=1}^{\infty} p_j^{c_j}$.

Inductive step: Assume the theorem is true for all $n \le m$ for some $m \in \mathbb{N}$ and consider $n=m+1$.

Existence of $c_1, c_2, c_3, \ldots$:

$\exists k \in \mathbb{N}\,(p_k \mid n)$. $\exists q \in \mathbb{N}\,(n=qp_k)$. By the inductive hypothesis, q has a prime factorization $q=\prod_{j=1}^{\infty} p_j^{a_j}$. $n=qp_k=\left(\prod_{j=1}^{k-1} p_j^{a_j}\right)p_k^{a_k+1}\left(\prod_{j=k+1}^{\infty} p_j^{a_j}\right)$

Uniqueness of $c_1, c_2, c_3, \ldots$:

Assume $n=\prod_{j=1}^{\infty} p_j^{c_j}$ and $n=\prod_{j=1}^{\infty} p_j^{c_j'}$ where $\forall j \in \mathbb{N}\,(c_j, c_j' \in \mathbb{N}\cup\{0\})$

$\exists k \in \mathbb{N}\,(p_k \mid n)$. By the prime property, $c_k \ge 1$ as otherwise $p_k \mid \prod_{j=1}^{\infty} p_j^{c_j}$ but p_k does not divide any term in this product.

Similarly, $c_k' \ge 1$. $\frac{n}{p_k}=\left(\prod_{j=1}^{k-1} p_j^{c_j}\right)p_k^{c_k-1}\left(\prod_{j=k+1}^{\infty} p_j^{c_j}\right)=\left(\prod_{j=1}^{\infty} p_j^{c_j'}\right)p_k^{c_k'-1}\left(\prod_{j=k+1}^{\infty} p_j^{c_j'}\right)$. By the inductive hypothesis, $\frac{n}{p_k}$ has a unique prime factorization so $\forall j \in \mathbb{N}\,(c_j'=c_j)$