Proofs by Induction

If we want to prove that $P(n)$ is true for all $n \in \mathbb{N}$, we can proceed as follows:

Base case: Prove $P(1)$ is true.

Inductive step: Prove $\forall k \in \mathbb{N}\,(P(k) \to P(k+1))$.

This is sufficient as:

$P(1)$ is true.

Since $P(1)$ is true, $P(2)$ is true.

Since $P(2)$ is true, $P(3)$ is true.

Since $P(3)$ is true, $P(4)$ is true.

etc.

This strategy is called a proof by induction.

Proofs by Induction

Example: Prove that $\forall n \in \mathbb{N} \left(\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \right)$

Base case: $n=1$

$$\sum_{j=1}^{1} j = 1 = \frac{1(1+1)}{2} \quad \checkmark$$

Inductive step: Assume $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$ (Inductive hypothesis)

$$\sum_{j=1}^{k+1} j = \left(\sum_{j=1}^{k} j \right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Thus, $\sum_{j=1}^{k} j = \frac{k(k+1)}{2} \rightarrow \sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2}$ $\checkmark$

Direct proof: Let $S = \sum_{j=1}^{n} j$.

$$S = 1 + 2 + \dots + (n-1) + n$$

$$S = n + (n-1) + \dots + 2 + 1$$

$$2S = \underbrace{(n+1) + (n+1) + \dots + (n+1) + (n+1)}_{n \text{ terms}}$$

$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

Proofs by Induction

The base case and inductive step can be adjusted as long as we don't have any circular logic and the base covers everything which is not covered by the inductive step.

Example: Consider a 2-player pebble game where we start with n pebbles, the players take turns taking 1, 2, or 3 pebbles, and whoever takes the last pebble wins.

Claim: The second player wins $\Leftrightarrow 4 | n$

Proof:

Base cases: $n = 1, 2, 3$, and 4.

If $n = 1, 2,$ or 3, the first player can take all the pebbles and win.

If $n = 4$, then if the first player takes $k \in \{1,2,3\}$ pebbles, the second player can take the remaining $4-k$ pebbles and win.

Inductive step: Assume that $\forall n \in [k]$, the second player wins $\Leftrightarrow 4|n$.

Consider $n = k+1$.

If $n = 4q+r$ where $r \in \{1,2,3\}$ then the first player can take r pebbles and the second player will be left with $4q$ pebbles and lose.

If $n = 4q$ then if the first player takes $a \in \{1,2,3\}$ pebbles, the second player can take $4-a$ pebbles. The first player will be left with $4(q-1)$ pebbles and lose.

Flawed Proof by Induction

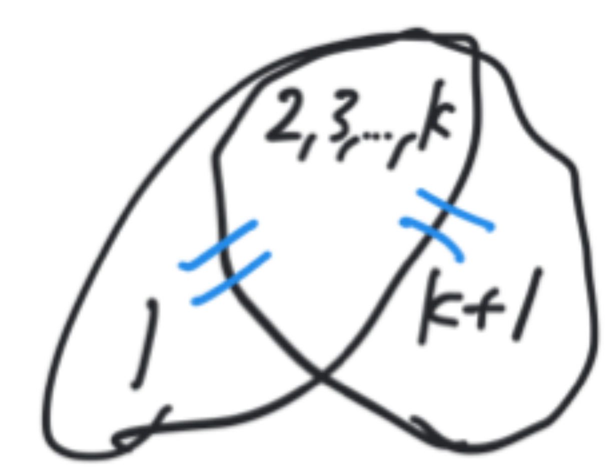
Claim: $\forall n \in \mathbb{N}$, any group of n horses is the same color.

"Proof":

Base case: $n=1$. Any group of 1 horse is the same color ✓

Inductive step: Assume any group of k horses has the same color.

Consider a group of $k+1$ horses. By the inductive hypothesis, the first k horses have the same color and the last k horses have the same color. Since these two subgroups have a horse in common, all $k+1$ horses must be the same color. false for $k=1$



Flaw: Inductive step fails for $k=1$.