Number of Positive Divisors

Q: Given a natural number n, how many positive divisors does n have?

Theorem: If the prime factorization of n is $n = \prod_{j=1}^{\infty} p_j^{c_j}$ then n has $\prod_{j=1}^{\infty} (c_j+1)$ positive divisors.

Example: $24 = 2^3 \cdot 3^1$ has $(3+1)(1+1) = 8$ positive divisors.

$\mathrm{Div}(24) \cap \mathbb{N} = \{1,2,3,4,6,8,12,24\}$

Proof:

Recall that $\mathrm{Div}(n) \cap \mathbb{N} = \{ \prod_{j=1}^{\infty} p_j^{a_j} : \forall j \in \mathbb{N},\ 0 \leq a_j \leq c_j \}$

Thus, choosing a positive divisor of n is equivalent to choosing an $a_j \in \{0,1,\ldots,c_j\}$.

$|\{0,1,\ldots,c_j\}| = c_j+1$ so the total number of choices is $\prod_{j=1}^{\infty}(c_j+1)$.

Sum of Positive Divisors

Q: Given a natural number n, what is the sum of its positive divisors?

Theorem: If the prime factorization of n is $n = \prod_{j=1}^{\infty} p_j^{c_j}$ then the sum of the positive divisors of n is $\prod_{j=1}^{\infty} \left(\sum_{a_j=0}^{c_j} p_j^{a_j} \right)$.

Example: The sum of the positive divisors of $12 = 2^2 \cdot 3^1$ is $(1+2+4)(1+3) = 7 \cdot 4 = 28$

$\mathrm{Div}(12) \cap \mathbb{N} = \{1, 2, 3, 4, 6, 12\}$ $\quad 1+2+3+4+6+12 = 28$

Idea: If we expand out $\prod_{j=1}^{\infty} \left(\sum_{a_j=0}^{c_j} p_j^{a_j} \right)$, this gives the positive divisors of n.

Example: For $n=12$, expanding out $(1+2+4)(1+3)$ gives the following diagram:

	$\times 1$	$\times 3$
$\times 1$	1	3
$\times 2$	2	6
$\times 4$	4	12