Divisibility

Definition: Given $d, n \in \mathbb{Z}$, we say that n is divisible by d (which we write as $d|n$) if $\exists q \in \mathbb{Z}(n = q \cdot d)$. If so, we say that d is a divisor of n and n is a multiple of d.

Example: $5|40$ because $40 = 8 \cdot 5$.

Example: $\forall n \in \mathbb{Z}(n|0)$ because $0 = 0 \cdot n$

Note: Using this definition, $0|0$. However, $\frac{0}{0}$ is undefined because there isn't a unique q such that $0 = q \cdot 0$.

Facts About Divisibility

Proposition: $\forall a, b \in \mathbb{Z} (a | a \cdot b \wedge b | a \cdot b)$

Proof:

To show $b | a \cdot b$, observe that $a \cdot b = q \cdot b$ where $q = a$.

To show $a | a \cdot b$, observe that $a \cdot b = q \cdot a$ where $q = b$.

Note: We are using that multiplication is <u>commutative</u>.

Proposition: For all $a, b, c \in \mathbb{Z}$, if $a | b$ and $b | c$ then $a | c$
(i.e. divisibility is <u>transitive</u>)

Proof:

Since $a | b$ and $b | c$, $\exists q_1, q_2 \in \mathbb{Z} (b = q_1 a \wedge c = q_2 b)$.

$c = q_2 b = q_2 (q_1 a) = (q_2 q_1) a$

Note: We are using that multiplication is <u>associative</u>.

Divisors

Definition: Given $n \in \mathbb{Z}$, we define $\mathrm{Div}(n) = \{d \in \mathbb{Z} : d \mid n\}$ to be the set of divisors of n.

Note: Remember that divisors can be both positive and negative.

Example: $\mathrm{Div}(6) = \{-6, -3, -2, -1, 1, 2, 3, 6\}$

Example: $\mathrm{Div}(0) = \mathbb{Z}$ because $\forall n \in \mathbb{Z}\ (n \mid 0)$