Universal and Existential Quantifiers

Existential quantifier $\exists$ = there exists

$\exists x \in S(P(x))$ says that there exists an $x \in S$ such that $P(x)$ is true.

Example: $\exists x \in \mathbb{Z}(x^2 - 4x + 3 < 0)$ says that there is an integer x such that $x^2 - 4x + 3 < 0$.

To prove a statement of the form $\exists x \in S(P(x))$ we just need to find an $x \in S$ such that $P(x)$ is true.

Example: $\exists x \in \mathbb{Z}(x^2 - 4x + 3 < 0)$

Proof: Take $x = 2$. $x^2 - 4x + 3 = 2^2 - 4 \cdot 2 + 3 = 4 - 8 + 3 = -1 < 0$

Universal and Existential Quantifiers

Universal quantifier $\forall$ = for all

$\forall x \in S(P(x))$ says that for all $x \in S$, $P(x)$ is true.

Example: $\forall x \in \mathbb{N}\,(x^2+3x+2 \notin \mathbb{P})$ says that for all natural numbers x, x^2+3x+2 is not prime.

To prove a statement of the form $\forall x \in S\,(P(x))$ we need to start with an arbitrary $x \in S$ and then prove $P(x)$.

Example: $\forall x \in \mathbb{N}\,(x^2+3x+2 \notin \mathbb{P})$

Proof:

Let x be a natural number. $x^2+3x+2 = (x+1)(x+2)$. Since $x \geq 1$, $x+1 \geq 2$ and $x+2 \geq 3$. Thus, $x^2+3x+2 \notin \mathbb{P}$.

More Universal and Existential Quantifier Examples

Definition: A Fermat number is a number of the form $F_n = 2^{(2^n)} + 1$ where $n \in \mathbb{N} \cup \{0\}$.

$F_0 = 2^1 + 1 = 3$, $F_1 = 2^2 + 1 = 5$, $F_2 = 2^4 + 1 = 17$, $F_3 = 2^8 + 1 = 257$, $F_4 = 2^{16} + 1 = 65537$

$F_0, F_1, F_2, F_3, F_4 \in \mathbb{P}$, so Fermat conjectured that all Fermat numbers are prime, i.e. $\forall n \in \mathbb{N} (F_n \in \mathbb{P})$.

This conjecture is false. $F_5 = 4294967297 \notin \mathbb{P}$.

In fact, we don't know if any other Fermat numbers are prime.

If we let Q be the proposition that there is another Fermat number which is prime, how can we write Q?

$Q: \exists n \in \mathbb{N} (n > 4 \wedge 2^{(2^n)} + 1 \in \mathbb{P})$

More Universal and Existential Quantifier Examples

Goldbach conjecture: Every even natural number which is larger than 2 is the sum of two prime numbers.

Examples: $4=2+2$ $6=3+3$ $8=3+5$ $10=5+5=3+7$
$12=5+7$, etc.

How can we write the Goldbach conjecture?

$G: \forall n \in \mathbb{N}\,(n \leq 1 \vee 2n$ is the sum of two primes$)$

$$G: \forall n \in \mathbb{N}\,(n \leq 1 \vee \exists p, q \in \mathbb{P}\,(2n = p+q))$$