Lectures 2 and 3: A Crash Course in Propositional Logic

Discrete Mathematics 27100 Winter 2022

Corresponding sections in Margaret Fleck's "Building Blocks for Theoretical Computer Science": Sections 2.1-2.9
Corresponding sections in Rosen's "Discrete Mathematics and Its Applications": Sections 1.1,1.2,1.3,1.6

Note: Not all of the material here is contained in these textbooks (or vice versa). In particular, the discussion of the law of the excluded middle, the principle of double negation, and contraposition is a special feature of these notes.

1 Propositions and logical operators

Definition 1.1. *In propositional logic, a proposition is a statement which is either true or false (though you might not know whether it is true or false).*

Definition 1.2. *An atomic proposition is a proposition which cannot be broken up into simpler parts.*

Example 1.3. *The following statements are atomic propositions.*

P: I am sleepy.

Q: I am hungry.

R: Pigs can fly.

S: Pigs can oink.

Remark 1.4. *In mathematics, Propositions are statements which are self-evident (at least to a mathematician) and thus need little if any proof. Note that Propositions in mathematics and propositions in logic are very different.*

We can build up propositions from other propositions using logical operators

Definition 1.5 (Logical operators)**.**

1. $\neg P$ *(not* P*) is true if and only if* P *is false.*
2. $P \vee Q$ *(*P *or* Q*) is true if and only if* P *is true or* Q *is true.*
3. $P \wedge Q$ *(*P *and* Q*) is true if and only if* P *and* Q *are both true*

4. $P \to Q$ (*P implies Q), which says that if P is true then Q is true as well, is true if and only if Q is true or P is false.*

5. $P \leftrightarrow Q$ (*P is equivalent to Q), is true if and only if P and Q are either both true or both false.*

Example 1.6. *If P,Q,R, and S are the statements given above then we can use the logical operators to build up the following more complex propositions:*

1. $\neg(P)$*: I am not sleepy.*

2. $Q \vee S$*: Either I am hungry or pigs can oink.*

3. $Q \wedge P$*: I am hungry and sleepy.*

4. $R \to P$*: If pigs can fly then I am sleepy.*

5. $S \to R$*: If pigs can oink then pigs can fly.*

Note that here the first and third statements may or may not be true, the second statement is true because pigs can oink, the fourth statement is also true because pigs can't fly, and the last statement is false because pigs can oink but they can't fly.

Remark 1.7. *When we hear an if then statement such as "If it is raining then I will use an umbrella", we expect the two parts of the statment to be related. However, in logic, the parts of a statement need not be related.*

Definition 1.8 (Order of operations)**.** *The order of operations for logical operators is as follows.*

1. $\neg$

2. $\wedge$

3. $\vee$

4. $\to$

5. $\leftrightarrow$

Example 1.9. *The proposition $P \to Q \wedge R$ says that "If P is true then both Q and R are true" rather than "R is true and P implies Q."*

Remark 1.10. *It is not universally agreed that $\wedge$ has higher precedence than $\vee$. When in doubt, use parentheses to make the order of operations clearer.*

2 Logical deduction

A fundamental type of question in propositional logic is as follows. Given axioms $A_1, A_2, \ldots, A_k$ (i.e. we start by assuming that the propositions $A_1, A_2, \ldots, A_k$ are true), can we deduce some other proposition P?

2.1 Truth Tables

One way to answer this type of question is with truth tables.

Definition 2.1 (Rough definition of truth tables)**.** *Given a proposition P which is built up from propositions $X_1, \ldots, X_k$, a truth table for P gives the value of P for every possible combination of which propositions X_i are true and which propositions X_i are false.*

Example 2.2. *The truth table for $\neg P$ (not P) is as follows:*

P	$\neg P$
F	T
T	F

Example 2.3. *The truth table for $P \vee Q$ (P or Q) is as follows:*

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Example 2.4. *The truth table for $P \wedge Q$ (P and Q) is as follows:*

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Example 2.5. *The truth table for $P \to Q$ (P implies Q) is as follows:*

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

Example 2.6. *Q: What is the truth table of $(P \to Q) \to R$?*

Answer:

P	Q	R	$(P \to Q) \to R$
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	F
T	T	T	T

2.2 Using truth tables

Truth tables are useful for checking whether a proposition P is a tautology, checking whether two propositions P and Q are equivalent, and checking whether we can deduce a proposition P from axioms $A_1, \ldots, A_k$.

Definition 2.7 (Tautologies)**.** *A proposition P is a tautology if it is always true regardless of whether the propositions it is built from are true or false.*

Example 2.8. *$P \to P$, i.e. "if P is true then P is true", is a tautology.*

Definition 2.9 (Equivalence)**.** *We say that P is equivalent to Q if Q is true if and only if P is true and this holds regardless of whether the propositions which P and Q are built from are true or false. In other words, $P \leftrightarrow Q$ is a tautology.*

Example 2.10. *$P \wedge Q$ is equivalent to $Q \wedge P$.*

Definition 2.11. *We say that we can deduce a proposition P from axioms $A_1, \ldots, A_k$ if whenever $A_1, \ldots, A_k$ are all true, P is true as well, and this holds regardless of whether the propositions which $A_1, \ldots, A_k$ and P are built from are true or false. In other words, $A_1 \wedge A_2 \wedge \ldots \wedge A_k \to P$ is a tautology.*

Using truth tables, we can easily check whether a proposition P is a tautology, whether two propositions P and Q are equivalent, and whether we can deduce a proposition P from axioms $A_1, \ldots, A_k$.

Proposition 2.12.

1. *P is a tautology if and only if every entry of the truth table for P is true.*
2. *P is equivalent to Q if and only if P and Q have the same truth tables.*
3. *We can deduce a proposition P from axioms $A_1, \ldots, A_k$ if and only if we have that for all of the rows where $A_1, \ldots, A_k$ are true, P is true as well.*

Example 2.13. *Q: Given that $P \to Q$, can we deduce that $\neg P \to \neg Q$?*

Answer: No. Looking at the truth tables for $P \to Q$ and $\neg P \to \neg Q$

P	Q	$P \to Q$	$\neg P \to \neg Q$
F	F	T	T
F	T	T	F
T	F	F	T
T	T	T	T

we see that if P is false and Q is true then P implies Q but $\neg P$ does not imply $\neg Q$.

Example 2.14. *Q: Given that $P \to Q$, can we deduce that $\neg Q \to \neg P$?*

Answer: Yes. Looking at the truth tables for $P \to Q$ and $\neg Q \to \neg P$

P	Q	$P \to Q$	$\neg Q \to \neg P$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

we see that the truth tables are the same. Thus, $P \to Q$ is in fact equivalent to $\neg Q \to \neg P$.

Example 2.15. *Q: Given that $\neg(P \wedge Q)$, can we deduce that $\neg P \wedge \neg Q$?*

Answer No. Looking at the truth tables for $\neg(P \wedge Q)$ and $\neg P \wedge \neg Q$

P	Q	$\neg(P \wedge Q)$	$\neg P \wedge \neg Q$
F	F	T	T
F	T	T	F
T	F	T	F
T	T	F	F

we see that if P is false and Q is true or P is true but Q is false then $\neg(P \wedge Q)$ is true but $\neg P \wedge \neg Q$ is false. Instead, $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$.

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
F	F	T	T
F	T	T	T
T	F	T	T
T	T	F	F

2.3 Inference rules

Q: Why don't we always use truth tables?

A: Truth tables can be extremely large. If a proposition P is built up from n propositions $X_1, \ldots, X_n$ then the truth table for P will have 2^n entries!

Since truth tables can be extremely large, it is generally more convenient to prove propositions using inference rules which allow us to deduce propositions from other propositions. This allows us to make progress step by step until we reach our desired conclusion.

One notation for this is as follows.

Definition 2.16. *We write that $P_1, \ldots, P_k \vdash Q$ if we are deducing Q from $P_1 \wedge \ldots \wedge P_k$ using an inference rule.*

Some fundamental inference rules are as follows:

1. Inference rules for AND statements:
   (a) $A, B \vdash A \wedge B$
   (b) $A \wedge B \vdash A$
   (c) $A \wedge B \vdash B$

2. Inference rules for IMPLIES statements

   (a) $A, A \to B \vdash B$ (modus ponens)

   (b) If we assume A and deduce B, then we have shown that $A \to B$.

   One way to write this is as follows. We first write "Assume A". We then make a series of logical deductions to deduce B. Once we are done, we can conclude that $A \to B$. For example, let's say we are given that $A \to B$ and $B \to C$ and we want to prove that $A \to C$. We can do this as follows:

   Assume A.

   $A, A \to B \vdash B$

   $B, B \to C \vdash C$

   Thus, $A \to C$.

   (c) $A \leftrightarrow B$ is shorthand for $(A \to B) \wedge (B \to A)$.

3. Inference rules for OR statements:

   (a) $A \vdash A \vee B$

   (b) $B \vdash A \vee B$

   (c) $A \vee B, A \to C, B \to C \vdash C$

4. Inference rules for negation, TRUE and FALSE (which we write as $\bot$ as it looks like an upside down T):

   (a) $\vdash TRUE$ (we always have TRUE as an axiom)

   (b) $\bot \vdash A$ (FALSE implies everything)

   (c) $\neg A$ is shorthand for $A \to \bot$

Example 2.17. *Let's say that we want to prove that $A \to \neg\neg A$. We can do this as follows:*

Assume A.

Assume $\neg A$.

$A, \neg A \vdash \bot$ *(recall that $\neg A$ is shorthand for $A \to \bot$)*

Thus, $\neg A \to \bot$, i.e. $\neg\neg A$.

Thus, $A \to \neg\neg A$.

Q: Are we missing any inference rules?

A: This question is surprisingly subtle! There are many more inference rules which we could list and we'll list several of them later. However, most of them can be deduced from the inference rules we already have, so they aren't really missing. However, the inference rules we have so far do not give us the full power of truth tables. If we only have the inference rules we've seen so far, this gives us a strange but perfectly valid system of logic called intuitionistic logic where propositions don't have to be either true or false.

In order to return to the more comfortable world of propositional logic where propositions must be either true or false, we need one of the following inference rules:

1. $\vdash P \vee \neg P$ (Law of the excluded middle)
2. $\neg\neg P \vdash P$ (Principle of double negation)
3. $(\neg P \rightarrow \neg Q) \vdash (Q \rightarrow P)$ (Contraposition)

2.4 Optional material: Equivalence of the Law of the Excluded Middle, the Principle of Double Negation, and Contraposition

Lemma 2.18. *Starting with any one of these inference rules and the previous inference rules, we can deduce the other two inference rules.*

Proof. We prove this by proving the following three statements.

1. Using the inference rule $(\neg P \rightarrow \neg Q) \vdash (Q \rightarrow P)$, we can deduce that $\neg\neg P \rightarrow P$.
2. Using the inference rule $\neg\neg P \vdash P$, we can deduce that $P \vee \neg P$.
3. Using the inference rule $\vdash P \vee \neg P$, we can deduce that $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$.

We can prove the first statement as follows. The idea is to apply the inference rule $(\neg P \rightarrow \neg Q) \vdash (Q \rightarrow P)$ with $Q = TRUE$.

Assume $\neg\neg P$.

> $\bot \rightarrow \neg TRUE$ (remember that FALSE implies everything)
>
> $\neg P \rightarrow \bot, \bot \rightarrow \neg TRUE \vdash \neg P \rightarrow \neg TRUE$ (recall that $\neg\neg P$ is shorthand for $\neg P \rightarrow \bot$)
>
> $\neg P \rightarrow \neg TRUE \vdash TRUE \rightarrow P$ (we obtained this by plugging in $Q = TRUE$ into the inference rule $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$)
>
> $\vdash TRUE$
>
> $TRUE, TRUE \rightarrow P \vdash P$

Thus, $\neg\neg P \rightarrow P$, as needed.

We can prove the second statement as follows. The idea is to apply the inference rule $\neg\neg Q \vdash Q$ with $Q = P \vee \neg P$.

Assume $\neg(P \vee \neg P)$.

> Assume P.

$P \vdash P \vee \neg P$

$P \vee \neg P, \neg(P \vee \neg P) \vdash \bot$ (remember that $\neg(P \vee \neg P)$ is shorthand for $P \vee \neg P \rightarrow \bot$)

Thus $P \rightarrow \bot$ (i.e. $\neg P$).

$\neg P \vdash P \vee \neg P$

$P \vee \neg P, \neg(P \vee \neg P) \vdash \bot$

Thus, $\neg(P \vee \neg P) \rightarrow \bot$ (i.e. $\neg\neg(P \vee \neg P)$).
$\neg\neg(P \vee \neg P) \vdash P \vee \neg P$ (we obtained this by plugging in $Q = P \vee \neg P$ into the inference rule $\neg\neg Q \vdash Q$)

We can prove the third statement as follows:

Assume $\neg P \rightarrow \neg Q$.

$\vdash P \vee \neg P$

Assume Q.

Assume P.

P

Thus, $P \rightarrow P$
Note: This deduction is a bit silly and it would be fine to instead directly write $\vdash P \rightarrow P$ as $P \rightarrow P$ is a tautology.

Assume $\neg P$.

$\neg P, \neg P \rightarrow \neg Q \vdash \neg Q$
$Q, \neg Q \vdash \bot$ (remember that $\neg Q$ is shorthand for $Q \rightarrow \bot$)
$\bot \vdash P$

Thus, $\neg P \rightarrow P$.

$P \vee \neg P, P \rightarrow P, \neg P \rightarrow P \vdash P$

Thus, $Q \rightarrow P$.

Thus, $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$. □

2.5 Additional inference rules and equivalences

We now describe some of the many additional inference rules and equivalences which can be derived from the inference rules of propositional logic.

1. Transitivity of $\rightarrow$ and $\leftrightarrow$:

   (a) $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

   (b) $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$

2. Commutativity of $\vee$ and $\wedge$:

   (a) $A \vee B \leftrightarrow B \vee A$

   (b) $A \wedge B \leftrightarrow B \wedge A$

3. Associativity of $\vee$ and $\wedge$:

   (a) $(A \vee B) \vee C \leftrightarrow A \vee (B \vee C)$

   (b) $(A \wedge B) \wedge C \leftrightarrow A \wedge (B \wedge C)$

4. Distributive laws:

   (a) $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$

   (b) $A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$

5. De Morgan's Laws:

   (a) $\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$

   (b) $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$

6. $(A \to B) \leftrightarrow \neg A \vee B$

7. $A \vee B, \neg A \vdash B$

8. Resolution rule: $A \vee B, \neg B \vee C \vdash A \vee C$

Remark 2.19. *While the resolution rule is simple, it can be surprisingly powerful. The resolution proof system is based on this resolution rule and SAT solvers use the resolution rule very effectively.*

We can use the equivalences above as follows. If $A \leftrightarrow B$ then we may freely replace A with B (or vice versa) in any proposition P. For example, given $A \vee \neg(B \vee C)$, since $\neg(B \vee C) \leftrightarrow \neg B \wedge \neg C$, we can deduce that $A \vee (\neg B \wedge \neg C)$. We can write such a step as follows.
$A \vee \neg(B \vee C), \neg(B \vee C) \leftrightarrow \neg B \wedge \neg C \vdash A \vee (\neg B \wedge \neg C)$

It would also be fine to make the deduction directly as long as we state which equivalence we are using to make the deduction. For example, it would be fine to write either of the following:

1. $A \vee \neg(B \vee C) \vdash A \vee (\neg B \wedge \neg C)$ (De Morgan's Law)

2. By De Morgan's Law, $A \vee \neg(B \vee C) \leftrightarrow A \vee (\neg B \wedge \neg C)$.

We can also turn inference rules into implications and add new inference rules as follows.

1. If we have an inference rule $P_1, P_2 \vdash Q$ then we can easily deduce that $P_1 \wedge P_2 \to Q$ as follows.

   Assume $P_1 \wedge P_2$.

   $P_1 \wedge P_2 \vdash P_1$

$P_1 \wedge P_2 \vdash P_2$

$P_1, P_2 \vdash Q$

Thus, $P_1 \wedge P_2 \rightarrow Q$.

Since we can always do this, if we have the inference rule $P_1 \wedge P_2 \rightarrow Q$, we can add the inference rule $\vdash P_1 \wedge P_2 \rightarrow Q$ as well. For example, since we have the inference rule $A \vdash A \vee B$, we may as well have the inference rule $\vdash A \rightarrow A \vee B$.

Following similar logic, if we have an inference rule $P_1, \ldots, P_k \vdash Q$ then we can add the inference rule $\vdash P_1 \wedge \ldots \wedge P_k \rightarrow Q$

2. Conversely, if we can prove that $(P_1 \wedge \ldots \wedge P_k) \rightarrow Q$ is a tautology then we can add $P_1, \ldots, P_k \vdash Q$ as a new inference rule. For example, the reason that we can add the inference rule $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$ is because we can show that $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$ is a tautology.

3 Negating propositions

Knowing how to negate statements is very useful for several reasons. If you are trying to show that a statement isn't always true, negating the statement tells you what you need to do to give a counterexample. If you are trying to prove the statement using a proof by contradiction, negating the statement is the first step.

For negating propositions, we can use the following rules:

1. $\neg\neg A \leftrightarrow A$ (principle of double negation)
2. $\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$ (De Morgan's Law)
3. $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$ (De Morgan's Law)
4. $\neg(A \rightarrow B) \leftrightarrow A \wedge \neg B$

Example 3.1. *Q: What is the negation of* $A \vee (\neg B \wedge C)$*?*

Answer: The negation of $A \vee (\neg B \wedge C)$ *is*

$$\neg(A \vee (\neg B \wedge C)) \leftrightarrow \neg A \wedge \neg(\neg B \wedge C) \leftrightarrow \neg A \wedge (\neg\neg B \vee \neg C) \leftrightarrow \neg A \wedge (B \vee \neg C)$$

In words, the negation of "Either A *is true or* B *is false and* C *is true" is "*A *is false and either* B *is true or* C *is false."*

Example 3.2. *Q: What is the negation of* $(A \vee B) \rightarrow C$*?*

Answer: The negation of $(A \vee B) \rightarrow C$ *is*

$$\neg((A \vee B) \rightarrow C) \leftrightarrow (A \vee B) \wedge \neg C$$

In words, the negation of "if either A *or* B *is true then* C *is true" is "*C *is false and either* A *or* B *is true."*

A Proofs of selected inference rules

In this section, we give proofs for De Morgan's Laws, the inference rule that $A \vee B, \neg A \vdash B$, and the resolution rule.

1. We can show that $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$ as follows.
   Assume $\neg(A \vee B)$.

   Assume A.

   $A \vdash A \vee B$

   $A \vee B, \neg(A \vee B) \vdash \bot$

   Thus, $A \rightarrow \bot$ (i.e. $\neg A$).

   Assume B.

   $B \vdash A \vee B$

   $A \vee B, \neg(A \vee B) \vdash \bot$

   Thus, $B \rightarrow \bot$ (i.e. $\neg B$).

   $\neg A, \neg B \vdash \neg A \wedge \neg B$

   Thus, $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$.

2. We can show that $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$ as follows.

   Assume $\neg A \wedge \neg B$

   $\neg A \wedge \neg B \vdash \neg A$

   $\neg A \wedge \neg B \vdash \neg B$

   Assume $A \vee B$

   $A \vee B, \neg A, \neg B \vdash \bot$

   Thus, $A \vee B \rightarrow \bot$ (i.e. $\neg(A \vee B)$)

   Thus, $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$.

3. We can show that $\neg A \vee \neg B \rightarrow \neg(A \wedge B)$ as follows.

   Assume $\neg A \vee \neg B$

   Assume $A \wedge B$.

   $A \wedge B \vdash A$

   $A \wedge B \vdash B$

   $A \vdash \neg\neg A$ (we showed earlier that $A \rightarrow \neg\neg A$ is a tautology)

   $B \vdash \neg\neg B$

   $\neg A \vee \neg B, \neg\neg A, \neg\neg B \vdash \bot$

   Thus, $A \wedge B \rightarrow \bot$, i.e. $\neg(A \wedge B)$

Thus, $\neg A \vee \neg B \to \neg(A \wedge B)$

4. We can show that $\neg(A \wedge B) \to \neg A \vee \neg B$ as follows.

   Assume $\neg(A \wedge B)$

   $\vdash A \vee \neg A$ (note that we are using the law of the excluded middle)
   $\vdash \neg A \to \neg A \vee \neg B$
   Assume A.
   $A, \neg A \vdash \perp$
   $\perp \vdash \neg A \vee \neg B$
   Thus, $A \to \neg A \vee \neg B$.
   $\vdash A \vee \neg A, A \to \neg A \vee \neg B, \neg A \to \neg A \vee \neg B \vdash \neg A \vee \neg B$

   Thus, $\neg(A \wedge B) \to \neg A \vee \neg B$.

5. To show that the inference rule $A \vee B, \neg A \vdash B$ is valid we need to show that given $A \vee B$ and $\neg A$ we can deduce B. We can do this as follows.

   $\vdash B \to B$
   Assume A.

   $A, \neg A \vdash \perp$
   $\perp \vdash B$

   Thus, $A \to B$.
   $A \vee B, A \to B, B \to B \vdash B$

6. To show that the resolution inference rule $A \vee B, \neg B \vee C \vdash A \vee C$ is valid we need to show that given $A \vee \neg B$ and $B \vee C$ we can deduce that $A \vee C$. We can do this as follows.

   $\vdash A \to A \vee C$
   Assume $\neg B$.

   $B \vee C, \neg B \vdash C$
   $C \vdash A \vee C$

   Thus, $\neg B \to A \vee C$.
   $A \vee \neg B, A \to A \vee C, \neg B \to A \vee C \vdash A \vee C$

Remark A.1. *Of the inference rules and equivalences listed here, the following statements rely on either the law of the excluded middle, the principle of double negation, or contraposition*

1. $(A \to B) \to \neg A \vee B$
2. $\neg(A \wedge B) \to \neg A \vee \neg B$

All of the other statements which we have seen so far (except for the law of the excluded middle, the principle of double negation, and contraposition themselves), can be proven without using the law of the excluded middle, the principle of double negation, or contraposition.

B Condensed Axioms

We can ask whether we really need all of the given axioms or whether the axioms can be condensed. In fact, the axioms can be condensed quite a bit. As shown by the logician Jan Lucasiewicz, the following inference rules together with the inference rule $A, A \rightarrow B \vdash B$ (modus ponens) are sufficient for expressing propositional logic.

1. $B \rightarrow (A \rightarrow B)$
2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
3. $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

Note: Here $A \vee B$ is expressed as $(A \rightarrow \perp) \rightarrow B$ and $A \wedge B$ is expressed as $\neg(\neg A \vee \neg B)$.

If you are up for a tough challenge, try to figure out how these inference rules capture all of the axioms we've seen for propositional logic. Warning: This is surprisingly tricky!