Law of the Excluded Middle

Surprisingly subtle question: Do we need more rules of deduction?

Actually, YES!

We need one of the following 3 rules:

1. Law of the Excluded Middle: $P \vee \neg P$
2. Principle of Double Negation: $\neg\neg P \rightarrow P$
3. Law of Contraposition: $P \rightarrow Q \leftrightarrow \neg Q \rightarrow \neg P$

Excellent, challenging exercise: Show that given any one of these rules, we can derive the other two rules.

Note: If we don't have these rules, this gives us a weird but perfectly valid system of logic called intuitionistic logic.

Law of the Excluded Middle Example

Example: $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$

Proving $\neg(A \wedge B) \rightarrow \neg A \vee \neg B$ requires the Law of the Excluded Middle!

Assume $\neg(A \wedge B)$

$A \vee \neg A$

Assume $\neg A$

$\neg A \vdash \neg A \vee \neg B$

Thus, $\neg A \rightarrow \neg A \vee \neg B$

Assume A

Assume B

$A, B \vdash A \wedge B$

$A \wedge B, A \wedge B \rightarrow \bot \vdash \bot$

Thus, $B \rightarrow \bot$ (i.e. $\neg B$)

$\neg B \vdash \neg A \vee \neg B$

Thus, $A \rightarrow \neg A \vee \neg B$

$A \vee \neg A, A \rightarrow \neg A \vee \neg B, \neg A \rightarrow \neg A \vee \neg B \vdash \neg A \vee \neg B$

Thus $\neg(A \wedge B) \rightarrow \neg A \vee \neg B$

Assume $\neg A \vee \neg B$

Assume $\neg A$

Assume $A \wedge B$

$A \wedge B \vdash A$

$A, A \rightarrow \bot \vdash \bot$

Thus, $A \wedge B \rightarrow \bot$ (i.e. $\neg(A \wedge B)$)

Thus, $\neg A \rightarrow \neg(A \wedge B)$

Assume $\neg B$

Assume $A \wedge B$

$A \wedge B \vdash B$

$B, B \rightarrow \bot \vdash \bot$

Thus, $A \wedge B \rightarrow \bot$ (i.e. $\neg(A \wedge B)$)

Thus, $\neg B \rightarrow \neg(A \wedge B)$

$\neg A \vee \neg B, \neg A \rightarrow \neg(A \wedge B), \neg B \rightarrow \neg(A \wedge B) \vdash \neg(A \wedge B)$

Thus, $\neg A \vee \neg B \rightarrow \neg(A \wedge B)$