Rules of Deduction

This lecture: Deduction rules for deriving propositions from other propositions.

Fundamental rules: How to introduce and use the logical connectives $\wedge$ AND / $\vee$ OR / $\rightarrow$ IMPLIES, and $\neg$ NOT.

Example:

Modus ponens: If A is true and A implies B then B is true.

Notation: Premise(s) $\vdash$ Conclusion

Modus ponens: $A, A \rightarrow B \vdash B$

Alternative notation:

$$\frac{\text{Premises}}{\text{Conclusion}}$$

Modus ponens: $\dfrac{A, A \rightarrow B}{B}$

Rules for AND

Introduce $\wedge$:

$A, B \vdash A \wedge B$

Using $\wedge$:

$A \wedge B \vdash A$

$A \wedge B \vdash B$

Example 1: Given $A \wedge B$, show $B \wedge A$ (i.e. $\wedge$ is commutative)

$A \wedge B \vdash A$

$A \wedge B \vdash B$

$B, A \vdash B \wedge A$

Example 2: Given $A \wedge (B \wedge C)$, show $(A \wedge B) \wedge C$ (i.e. $\wedge$ is associative)

$A \wedge (B \wedge C) \vdash A$

$A \wedge (B \wedge C) \vdash B \wedge C$

$B \wedge C \vdash B$

$B \wedge C \vdash C$

$A, B \vdash A \wedge B$

$A \wedge B, C \vdash (A \wedge B) \wedge C$

Rules for IMPLIES

Using $\to$:

Modus ponens: $A, A \to B \vdash B$

Introducing $\to$:

To show $A \to B$, we must assume A and deduce B.

Structure:

Assume A

Proof that B is true assuming A is true.

Thus, $A \to B$

Example: Given $A \to B, B \to C$, show $A \to C$ (i.e. $\to$ is transitive)

Assume A

$A, A \to B \vdash B$

$B, B \to C \vdash C$

Thus, $A \to C$

Rules for OR

Introducing $\vee$:

$A \vdash A \vee B$

$B \vdash A \vee B$

Using $\vee$:

$A \vee B,\ A \rightarrow C,\ B \rightarrow C \vdash C$

Example: Given $A \vee B$, show $B \vee A$ (i.e. $\vee$ is commutative)

Assume A

$A \vdash B \vee A$

Thus, $A \rightarrow B \vee A$

Assume B

$B \vdash B \vee A$

Thus, $B \rightarrow B \vee A$

$A \vee B,\ A \rightarrow B \vee A,\ B \rightarrow B \vee A \vdash B \vee A$

Rules for OR

Example: Given $A \vee (B \vee C)$, show $(A \vee B) \vee C$ (i.e. $\vee$ is associative)

Assume A

$A \vdash A \vee B$

$A \vee B \vdash (A \vee B) \vee C$

Thus, $A \to (A \vee B) \vee C$

Assume B

$B \vdash A \vee B$

$A \vee B \vdash (A \vee B) \vee C$

Thus, $B \to (A \vee B) \vee C$

Assume C

$C \vdash (A \vee B) \vee C$

Thus, $C \to (A \vee B) \vee C$

Assume $B \vee C$

$B \vee C,\ B \to (A \vee B) \vee C,\ C \to (A \vee B) \vee C \vdash (A \vee B) \vee C$

Thus, $B \vee C \to (A \vee B) \vee C$

$A \vee (B \vee C),\ A \to (A \vee B) \vee C,\ B \vee C \to (A \vee B) \vee C \vdash (A \vee B) \vee C$

Rules for NOT

$\neg A$ is shorthand for $A \to \bot$

$\bot$ = FALSE

Using $\bot$:

$\bot \vdash C$ ← arbitrary conclusion

Example: Given $A \lor \neg B$, B, show A

Assume A

A

Thus $A \to A$

Assume $\neg B$

$B, B \to \bot \vdash \bot$

$\bot \vdash A$

Thus, $\neg B \to A$

$A \lor \neg B, A \to A, \neg B \to A \vdash A$

Example: De Morgan's Law

De Morgan's Laws:

$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$

$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$

Let's prove $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$

Assume $\neg(A \vee B)$

Assume A

$A \vdash A \vee B$

$A \vee B, A \vee B \to \bot \vdash \bot$

Thus, $A \to \bot$ (i.e. $\neg A$)

Assume B

$B \vdash A \vee B$

$A \vee B, A \vee B \to \bot \vdash \bot$

Thus, $B \to \bot$ (i.e. $\neg B$)

$\neg A, \neg B \vdash \neg A \wedge \neg B$

Thus, $\neg(A \vee B) \to \neg A \wedge \neg B$

Assume $\neg A \wedge \neg B$

$\neg A \wedge \neg B \vdash \neg A$

$\neg A \wedge \neg B \vdash \neg B$

Assume $A \vee B$

$A \vee B, A \to \bot, B \to \bot \vdash \bot$

Thus, $A \vee B \to \bot$ (i.e. $\neg(A \vee B)$)

Thus, $\neg A \wedge \neg B \to \neg(A \vee B)$