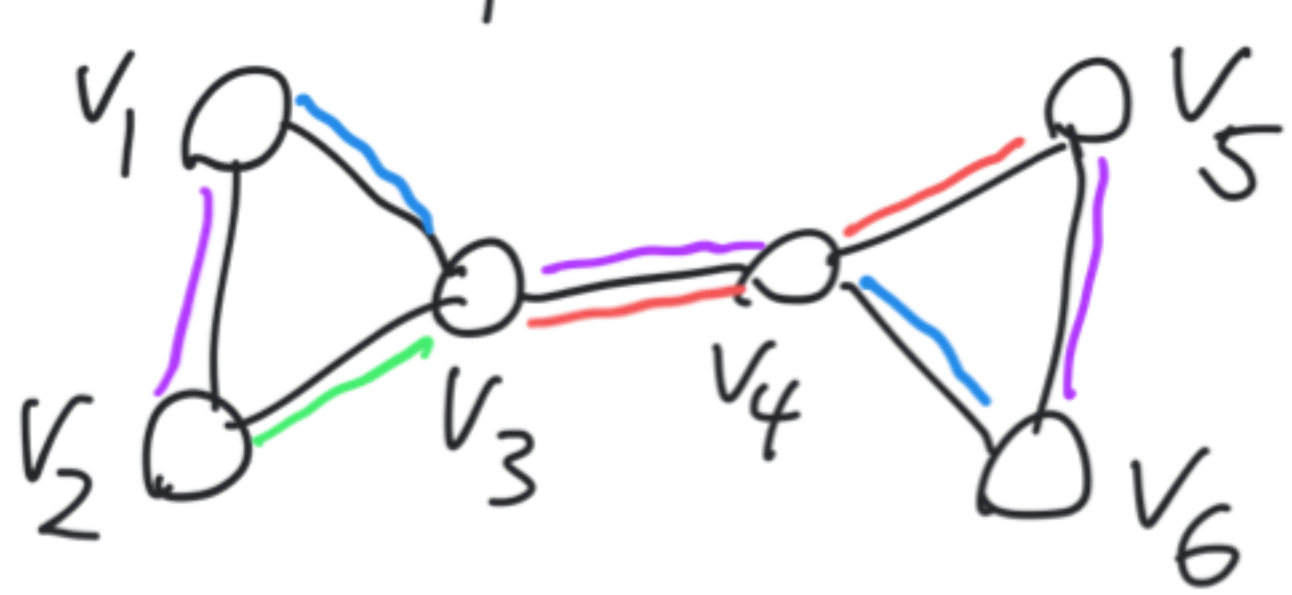
Matchings

Definition: Given an undirected graph G, a **matching** is a set of edges $M \subseteq E(G)$ such that no two edges in M share an endpoint.

Examples:



$M_1 = \{\{v_1, v_3\}, \{v_4, v_6\}\}$

$M_2 = \{\{v_2, v_3\}\}$

$M_3 = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}\}$

are all matchings.

$E = \{\{v_3, v_4\}, \{v_4, v_5\}\}$ is not a matching because the two edges share the endpoint v_4.

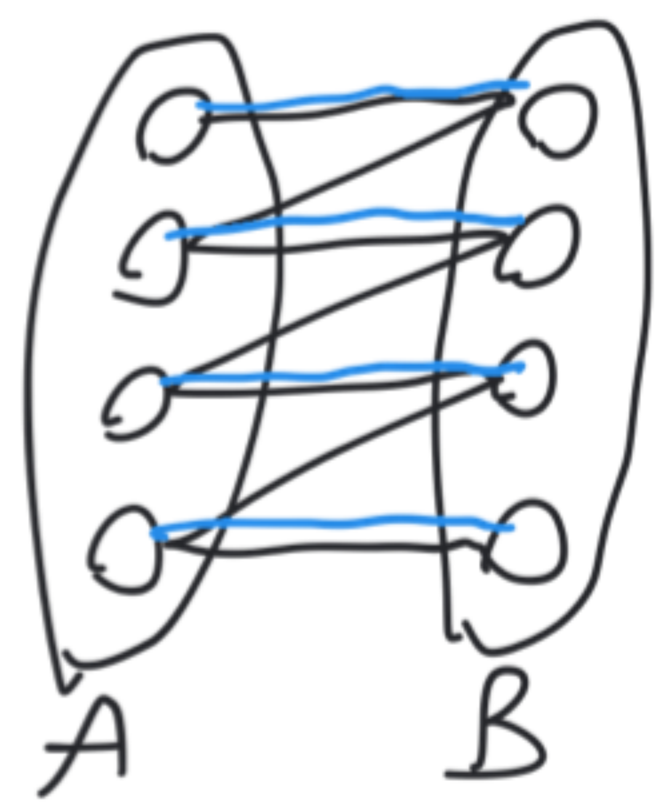
Maximum matching problem: What is the largest matching in a graph G?

Note: This can be solved using Jack Edmond's Blossom algorithm.

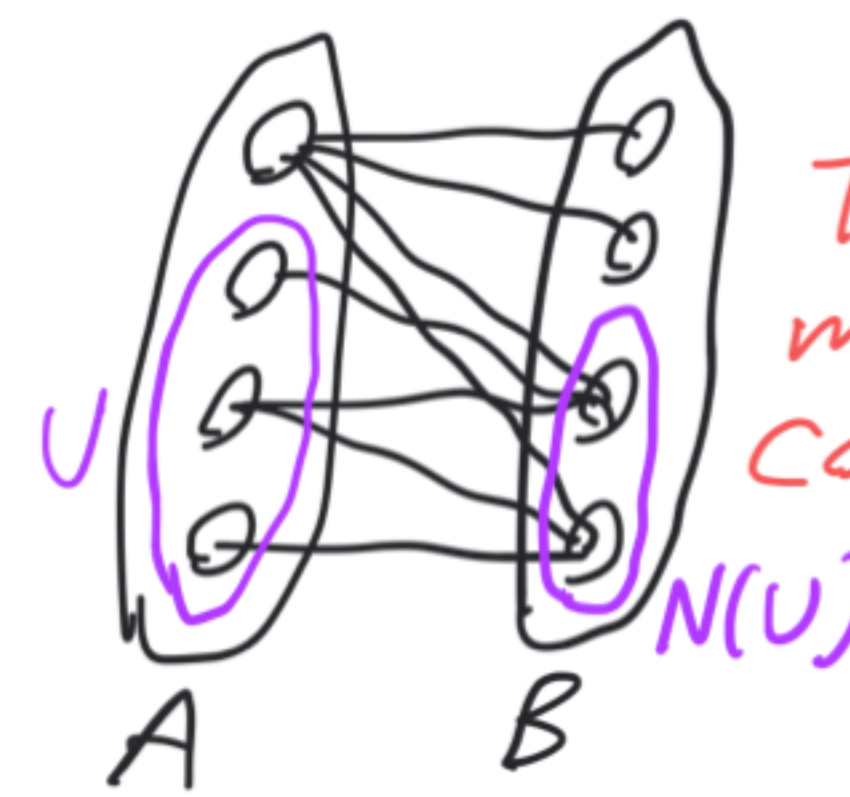
Matchings in Bipartite Graphs

Q: Given a bipartite graph G where $V(G)$ are partitioned into sets A and B, is there a matching M covering all of the vertices in A?

Examples:



Matching M covering A



There is no matching covering A.

$|N(U)| < |U|$

Q: What prevents us from having such a matching M?

Definition: Given a set of vertices U define the neighborhood of U to be the set

$$N(U) = \{v \in V(G) : \exists u \in U (\{u,v\} \in E(G))\}$$

Proposition: If $\exists U \subseteq A \, (|N(U)| < |U|)$ then there is no matching covering A.

Q: Is the only kind of obstacle? YES!

Hall's Theorem

Hall's Theorem: Let G be a bipartite graph and let A and B be the parts of G. There is a matching M covering A if and only if $\forall U \subseteq A\ (|N(U)| \geq |U|)$.

Proof: By induction on $|A|$.

Base case: $|A| = 0$. This case is trivial.

Inductive step: Assume the result is true whenever $|A| \leq k$ and consider the case when $|A| = k+1$.

Case 1: If $\exists U \subseteq A\ (|N(U)| < |U|)$ then no such M exists.

Case 2: If $|N(U)| > |U|$ whenever $U \subseteq A$, $U \neq \emptyset$ and $U \neq A$ then take an arbitrary edge $\{u, v\}$ where $u \in A$ and $v \in B$.
Now if $U' \subseteq A \setminus \{u\}$ and $U' \neq \emptyset$ then $|N(U') \setminus \{v\}| \geq |N(U')| - 1 \geq |U'|$ as $|N(U')| > |U'|$.
By the inductive hypothesis, there is a matching M' between $A \setminus \{u\}$ and $B \setminus \{v\}$ which covers $A \setminus \{u\}$.
$M = M' \cup \{u, v\}$.

Hall's Theorem

Case 3: $\forall U \subseteq A\ (|N(U)| \geq |U|)$ and
$\exists U \subseteq A\ (U \neq \emptyset \wedge U \neq A \wedge |N(U)| = |U|)$

Idea: We must match the vertices in U with the vertices in $N(U)$.

Goal: Show that we can do this and match the vertices in $A \setminus U$ to vertices in $B \setminus N(U)$.

Observation 1: For any $U' \subseteq U$, $N(U') \subseteq N(U)$ and $|N(U')| \geq |U'|$ so by the inductive hypothesis, there is a matching M_1 between U and $N(U)$ covering all vertices of U.

Observation 2: For any $U' \subseteq A \setminus U$, $|N(U') \setminus N(U)| \geq |U'|$ as otherwise
$$|N(U \cup U')| = |N(U)| + |N(U') \setminus N(U)| < |U| + |U'| = |U \cup U'|.$$

By the inductive hypothesis, there is a matching M_2 between $A \setminus U$ and $B \setminus N(U)$ covering all vertices of $A \setminus U$.

Take $M = M_1 \cup M_2$.