Graph Theory Part 2

Discrete Mathematics 27100 Winter 2022

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1 Bipartite Graphs

Definition 1.1. *We say that an undirected graph G is bipartite if there is a partition (A, B) of the vertices $V(G)$ of G such that $\forall u \in A, v \in A \setminus \{u\} (\{u, v\} \notin E(G))$ and $\forall u \in B, v \in B \setminus \{u\} (\{u, v\} \notin E(G))$.*

Remark 1.2. *We can think of the partition (A, B) as assigning two colors A and B to the vertices $V(G)$ so that no two adjacent vertices have the same color.*

Theorem 1.3. *An undirected graph G is bipartite if and only if G does not contain any cycles of odd length.*

Proof. If G has an cycle $C = \{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{l-1}, v_0\}$ of odd length then G cannot be bipartite. To see this, assume that there is a partition (A, B) of the vertices of G such that $\forall u \in A, v \in A \setminus \{u\} (\{u, v\} \notin E(G))$ and $\forall u \in B, v \in B \setminus \{u\} (\{u, v\} \notin E(G))$.

Without loss of generality, $v_0 \in A$. If $v_0 \in A$ then v_1 must be in B. If $v_1 \in B$ then v_2 must be in A. Continuing in this way, we must have that whenever j is even, v_j is in A and whenever j is odd, v_j is in B. However, this implies that $v_0, v_{l-1} \in A$ which is a contradiction as $\{v_0, v_{l-1}\} \in E(G)$.

If G has no cycles of odd length then we can construct A and B as follows.

1. Choose a vertex u_i in each connected component of G and put each u_i in A.

2. For each vertex $v \in V(G)$, if v is in the same connected component as u_i then put v in A if there is a walk of even length from u_i to v and put v in B if there is a walk of odd length from u_i to v.

We need to check two things.

1. We need to check that this is well-defined. In other words, we need to check that every vertex $v \in V(G)$ is put into A or B and no vertex v is put into both A and B.

2. We need to check that $\forall u \in A, v \in A \setminus \{u\} (\{u, v\} \notin E(G))$ and $\forall u \in B, v \in B \setminus \{u\} (\{u, v\} \notin E(G))$.

For the first part, observe that every $v \in V(G)$ is in some connected component of G so every $v \in V(G)$ is put into either A or B. Assume that some $v \in V(G)$ is put into both A and B. If so, there is a walk W_1 from u_i to v of even length and a walk W_2 from u_j to v of odd length for some u_i and u_j. Taking W_1 and then taking W_2 in reverse gives a walk W' of odd length which starts at u_i and ends at u_j. This implies that u_i and u_j are in the same connected component, so $u_i = u_j$ and W' starts and ends at the same vertex. By the following lemma, this implies that G contains a cycle of odd length, which is a contradiction.

Lemma 1.4. *If G has a walk of odd length that starts and ends at the same vertex then G has a cycle of odd length.*

Proof. Let $W = \{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{l-1}, v_l\}$ be a walk such that

1. l is odd, $v_l = v_0$, and $\forall j \in [l] \left(\{v_{j-1}, v_j\} \in E(G)\right)$ (W is a walk of odd length in G starting and ending at the same vertex)
2. There is no walk W' in G which has odd length, starts and ends at the same vertex, and is shorter than W.

We claim that W must be a cycle. To see this, assume that $v_i = v_j$ for some i, j such that $0 \leq i < j \leq l$. If $i + (l - j)$ is odd then $W' = \{v_0, v_1\}, \ldots, \{v_{i-1}, v_i\}, \{v_j, v_{j+1}\}, \ldots, \{v_{l-1}, v_l\}$ is a shorter walk which also has odd length and starts and ends at the same vertex, which contradicts our choice of W. If $i + (l - j)$ is even then $j - i$ is odd and $W' = \{v_i, v_{i+1}\}, \ldots, \{v_{j-1}, v_j\}$ is a shorter walk which also has odd length and starts and ends at the same vertex, which contradicts our choice of W. Thus, W must be a cycle, as needed. □

For the second statement, assume that there there exist two vertices $v, w \in A$ such that $\{v, w\} \in E(G)$. Since $v, w \in A$, there is a walk W_1 from u_i to v of even length and a walk W_2 from u_j to w of even length for some u_i and u_j. Taking W_1, taking the edge $\{v, w\}$, and then taking W_2 in reverse gives a walk W' of odd length which starts at u_i and ends at u_j. Again, this implies that u_i and u_j are in the same connected component, so $u_i = u_j$ and W' starts and ends at the same vertex. Again, using Lemma 1.4, this implies that G has a cycle of odd length which is a contradiction.

Following similar logic, $\forall u \in B, v \in B \setminus \{u\} \left(\{u, v\} \notin E(G)\right)$. □

2 Matchings on Bipartite Graphs and Hall's Theorem

Definition 2.1. *A matching on an undirected graph G is a set of edges $M = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ in G such that no two edges have a vertex in common.*

Definition 2.2. *Let G be an undirected graph. Given a vertex $u \in V(G)$, we define the neighbors $N(u)$ of u to be the set $N(u) = \{v \in V(G) : \{u, v\} \in E(G)\}$ of vertices in G which are adjacent to u.*

Similarly, given a set $U \subseteq V(G)$, we define the neighbors $N(U)$ of U to be $N(U) = \{v \in V(G) : \exists u \in U(\{u, v\} \in E(G))\}$.

Theorem 2.3 (Hall's Theorem)**.** *Let G be a bipartite graph and let A and B be the two sides of G. There exists a matching of size $|A|$ between A and B if and only if for every $U \subseteq A$, $|N(U)| \geq |U|$.*

Proof. Let $m = |A|$. If there is a matching $M = \{\{a_j, b_{i_j}\} : j \in [m]\}$ which covers all of the vertices of A then for any $J \subseteq [m]$, $\{b_{i_j} : j \in J\} \subseteq N(\{a_j : j \in J\})$. Since the vertices $b_{i_1}, \ldots, b_{i_m}$ are all distinct, $\forall J \subseteq [m]\, (|N(\{a_j : j \in J\})| \geq |J|)$.

We now need to show that if $\forall U \subseteq A(|N(U)| \geq |U|)$ then G has a matching which covers A. We prove this by induction on $|A| = m$.

The base case $m = 1$ is trivial as if $A = \{a_1\}$ and $|N(A)| \geq 1$ then a_1 is adjacent to at least one vertex of B.

For the inductive step, assume the result is true for all $m \leq k$ and consider the case when $m = k + 1$. There are two cases to consider:

1. $\exists U \subseteq A : 0 < |U| < |A|, |N(U)| = |U|$.
2. $\nexists U \subseteq A : 0 < |U| < |A|, |N(U)| = |U|$.

If $\exists U \subseteq A\, (0 < |U| < |A|, |N(U)| = |U|)$ then observe that for all $U' \subseteq U$, $N(U') \subseteq U$ and $N(U') \geq |U'|$. By the inductive hypothesis, there is a matching of size $|U|$ between U and $N(U)$. Also observe that for any $V \subseteq A \setminus U$,

$$|N(U \cup V)| = |N(V) \setminus N(U)| + |N(U)| \geq |U \cup V| = |U| \cup |V|$$

Since $|N(U)| = |U|$, this implies that $|N(V) \setminus N(U)| \geq |V|$. By the inductive hypothesis, there is a matching of size $|A| - |U|$ between $A \setminus U$ and $B \setminus N(U)$. Taking the union of these two matchings gives a matching of size $|A|$ between A and B, as needed.

If $\nexists U \subseteq A\, (0 < |U| < |A|, |N(U)| = |U|)$ then for all $U \subseteq A$ such that $0 < |U| < |A|$, $|N(U)| \geq |U| + 1$. If so, consider an aribtrariy edge $\{u, v\}$ where $u \in A$ and $v \in B$. Observe that for any $U \subseteq A \setminus \{u\}$, $|N(U) \setminus \{v\}| \geq |N(U)| - 1 \geq |U|$. By the inductive hypothesis, this implies that there is a matching of size $|A| - 1$ between $A \setminus \{u\}$ and $B \setminus \{v\}$. Adding the edge $\{u, v\}$ to this matching gives a matching of size $|A|$ between A and B, as needed. □

Definition 2.4. *A vertex cover of an undirected graph G is a set of vertices V such that for all edges $e = (u, v) \in E(G)$, either $u \in V$ or $v \in V$ (or both).*

Corollary 2.5 (König's Theorem)**.** *For all bipartite graphs G, the maximum size of a matching M in G is equal to the minimum size of a vertex cover of G.*

Proof. Let k be the size of a maximum matching of G and let $M = \{\{a_1, b_1\}, \ldots, \{a_k, b_k\}\}$ be a matching of size k in G. For any vertex cover V of G, for any $j \in [k]$, either $a_j \in V$ or $b_j \in V$, so we must have that $|V| \geq k$. Thus, the minimum size of a vertex cover of G is greater than or equal to the maximum size of a matching in G.

Now let V be a vertex cover of G of minimum size. We need to construct a matching M in G with $|V|$ edges. To do this, decompose V as $V_1 \cup V_2$ where $V_1 = V \cap A$ and $V_2 = V \cap B$.

We now show that for all $U \subseteq V_1$, $|N(U) \cap (B \setminus V_2)| \geq |U|$. To see this, assume that $|N(U) \cap (B \setminus V_2)| < |U|$ for some $U \subseteq V_1$. If so, observe that $V' = (V_1 \setminus U) \cup V_2 \cup (N(U) \cap (B \setminus V_2))$ is also a vertex cover of G. To see this, consider an edge $e = \{u, v\} \in E(G)$ where $u \in A$ and $v \in B$.

1. If $u \notin V_1$ then $v \in V_2 \subseteq V'$ (as otherwise V would not be a vertex cover of G).

2. If $u \in V_1 \setminus U$ then $u \in V'$.

3. $N(U) \subseteq V'$ so if $u \in U$ then $v \in V'$.

Now observe that since V is a vertex cover of minimum size, $|V'| = |V_1| - |U| + |V_2| + |N(U) \cap (B \setminus V_2)| \geq |V| = |V_1| + |V_2|$. Thus, $|N(U) \cap (B \setminus V_2)| \geq |U|$, as needed.

By Hall's theorem, there is a matching of size $|V_1|$ between $|V_1|$ and $B \setminus V_2$. Following similar logic, there is also a matching of size $|V_2|$ between $A \setminus V_1$ and V_2. Combining these matchings gives a matching between A and B of size $|V_1| + |V_2| = |V|$. Thus, the maximum size of a matching in G is equal to the minimum size of a vertex cover of G. $\square$