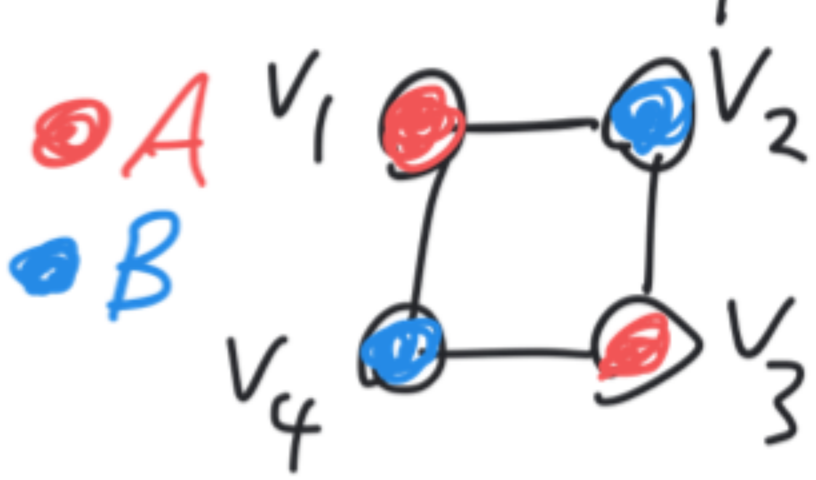
Bipartite Graphs

Definition: We say that an undirected graph G is bipartite if there is a partition of $V(G)$ into two sets A and B such that no edge of G has both endpoints in A or both endpoints in B (i.e. $\forall e=\{u,v\}\in E(G)\ ((u\in A \land v\in B) \lor (u\in B \land v\in A))$).

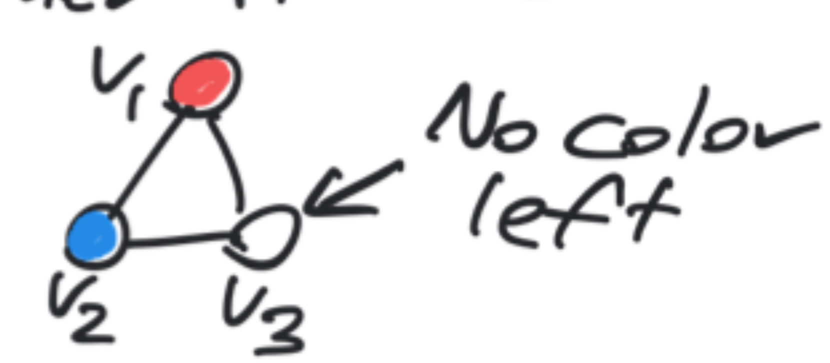
Examples:



is bipartite as if we take $A=\{v_1,v_3\}$ and $B=\{v_2,v_4\}$ then there are no edges with both endpoints in A or both endpoints in B.

Equivalently, we can view this as giving each vertex of G one of two colors so that no two adjacent vertices have the same color.

is not bipartite as for any partition of the vertices into two sets A and B, either A or B must have at least two vertices and thus an edge.



Bipartite Graphs

Q: Which graphs G are bipartite?

Theorem: An undirected graph G is bipartite if and only if G does not contain a cycle of odd length.

Proof:

G has a cycle of odd length $\rightarrow$ G is not bipartite:

Assume G has a cycle $C = \{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{\ell-1}, v_0\}$ where ℓ is odd and there is a partition of $V(G)$ into two sets A and B such that no edge of G has both endpoints in A or both endpoints in B. Without loss of generality, $v_0 \in A$.

Lemma: For all $j \in \{0, 1, \ldots, \ell-1\}$, if j is even then $v_j \in A$ and if j is odd then $v_j \in B$.

Proof: By induction.

Base case: $j = 0$. $v_0 \in A$.

Inductive step: Assume the statement is true for $j = k$ and consider $j = k+1$.

If $j = k+1$ is even then by the inductive hypothesis, $v_k \in B$ so $v_{k+1} \in A$.

If $j = k+1$ is odd then by the inductive hypothesis, $v_k \in A$ so $v_{k+1} \in B$.

Now $v_0, v_{\ell-1} \in A$ and $\{v_0, v_{\ell-1}\} \in E(G)$. Contradiction.

Picture:

Bipartite Graphs

G has no cycle of odd length $\rightarrow$ G is bipartite:

Key Lemma: If G contains a walk W of odd length from a vertex v to itself then G contains a cycle of odd length.

Proof: Let $C = \{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{l-1}, v_0\}$ be a minimum length walk from a vertex to itself of odd length.

Claim: C is a cycle.

To see this, assume $v_i = v_j$ for some i, j such that $0 \le i < j \le l-1$.
If so, $W_1 = \{v_0, v_1\}, \ldots, \{v_{i-1}, v_i\}, \{v_j, v_{j+1}\}, \ldots, \{v_{l-1}, v_0\}$ and $W_2 = \{v_i, v_{i+1}\}, \ldots, \{v_{j-1}, v_j\}$ are two shorter walks from a vertex to itself, one of which must have odd length. Contradiction.

Picture: 

Corollary: If G has no cycles of odd length and $v, w \in V(G)$ then there cannot be both a walk of even length from v to w and a walk of odd length from v to w.

Bipartite Graphs

G has no cycle of odd length → G is bipartite:
We can consider each connected component of G individually so it is sufficient to consider the case when G is connected.

To partition $V(G)$ into A and B:

1. Choose a vertex $v \in V(G)$.
2. Let $A = \{w \in V(G) : \text{there is a walk of even length from } v \text{ to } w\}$
   and let $B = \{w \in V(G) : \text{there is a walk of odd length from } v \text{ to } w\}$

Claim: No edge has both endpoints in A or both endpoints in B.

To see this, assume $w, w' \in A$ and $\{w, w'\} \in E(G)$. If so, there is a walk of even length from v to w' (as $w' \in A$) and a walk of odd length from v to w' (we can take the walk from v to w and then the edge $\{w, w'\}$). Contradiction.

Picture:
- even length walk from v
- odd length walk from v