Walks, Paths, and Cycles

Definition: A walk W in a graph G is a sequence of edges $\{v_0,v_1\}, \{v_1,v_2\}, \dots, \{v_{l-1},v_l\}$ (or $(v_0,v_1),(v_1,v_2), \dots, (v_{l-1},v_l)$ for a directed graph) . The length of W is the number of edges l in W.

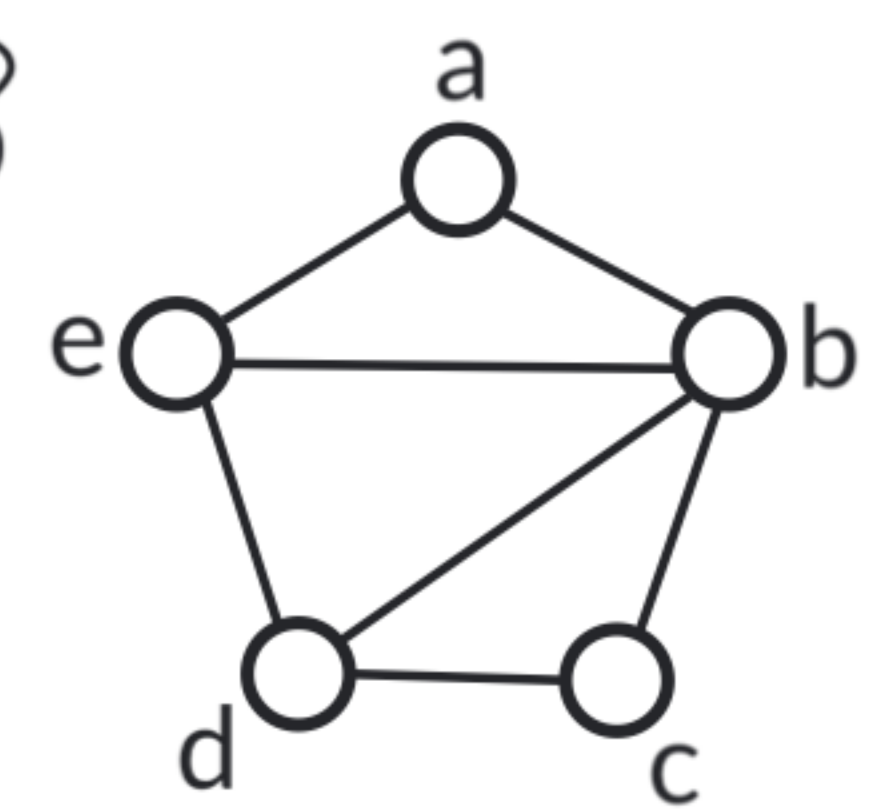
Definition: A path is a walk where all of the vertices $v_0, v_1, \dots, v_l$ are distinct.

Definition: A cycle is a walk where $v_l=v_0$ and $v_0, v_1, \dots, v_{l-1}$ are distinct. In other words, a cycle is a walk which starts and ends at the same vertex but has no other repeated vertices.

Example: $W = \{a,b\}, \{b,d\}, \{d,e\}, \{e,b\}, \{b,c\}$ is a walk from a to c of length 5.

Example: $P = \{a,b\}, \{b,e\}, \{e,d\}, \{d,c\}$ is a path from a to c of length 4.

Example: $C = \{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}, \{e,a\}$ is a cycle of length 5.



G

Facts About Walks and Paths

Proposition: If $s, m, t \in V(G)$, we have a walk W_1 from s to m of length ℓ_1, and we have a walk W_2 from m to t of length ℓ_2 then composing W_1 and W_2 gives a walk from s to t of length $\ell_1+\ell_2$.

Proof: Writing

$W_1 = \{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{\ell_1-1}, v_{\ell_1}\}$ where $v_0 = s$ and $v_{\ell_1} = m$

$W_2 = \{v'_0, v'_1\}, \{v'_1, v'_2\}, \ldots, \{v'_{\ell_2-1}, v'_{\ell_2}\}$ where $v'_0 = m$ and $v'_{\ell_2} = t$

and taking $v_{\ell_1+j} = v'_j$ for $j \in \{0, 1, \ldots, \ell_2\}$, composing the walks W_1 and W_2 gives the walk

$$\{v_0, v_1\}, \{v_1, v_2\}, \ldots \{v_{\ell_1-1}, v_{\ell_1}\}, \underbrace{\{v'_0, v'_1\}, \ldots, \{v'_{\ell_2-1}, v'_{\ell_2}\}}_{= \{v_{\ell_1}, v_{\ell_1+1}\}, \ldots, \{v_{\ell_1+\ell_2-1}, v_{\ell_1+\ell_2}\}}$$

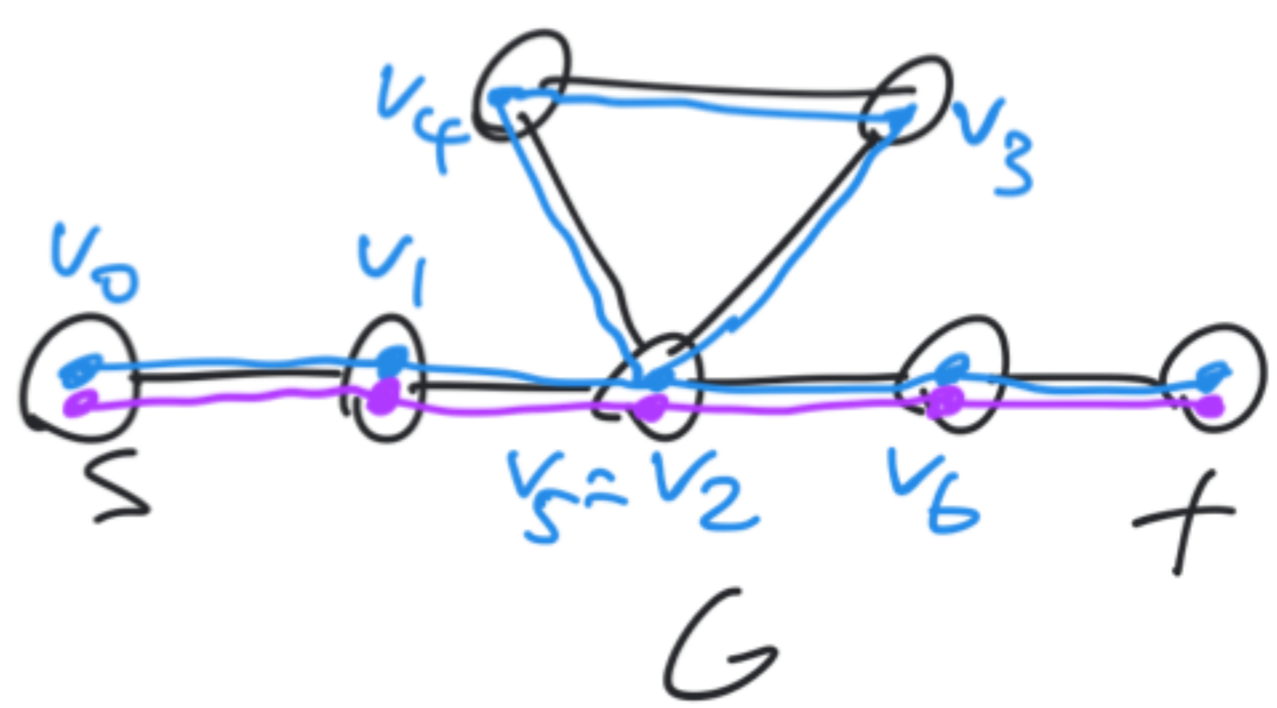
which is a walk from s to t of length $\ell_1+\ell_2$ as $v_0 = s$ and $v_{\ell_1+\ell_2} = v'_{\ell_2} = t$.

Facts About Walks and Paths

Lemma: If $s,t \in V(G)$ and P is a walk from s to t in G of <u>minimum length</u> then P is a path.

Proof: Write $P = \{v_0, v_1\}, \ldots, \{v_{\ell-1}, v_\ell\}$ where $v_0 = s$ and $v_\ell = t$. Assume P is not a path. Then $\exists\, j, j' \in \{0, 1, \ldots, \ell\} (j < j' \wedge v_{j'} = v_j)$. If so, then $\{v_0, v_1\}, \ldots, \{v_{j-1}, v_j\}, \{v_{j'}, v_{j'+1}\}, \ldots, \{v_{\ell-1}, v_\ell\}$ is a shorter walk from s to t. Contradiction.

Picture:



Walk P

Shorter walk

Facts About Walks and Paths

Corollary 1: There is a path from s to t in a graph G if and only if there is a walk from s to t in G.

Proof:

Path $\rightarrow$ Walk: Any path is a walk

Walk $\rightarrow$ Path: The shortest walk from s to t is a path.

Corollary 2: If there is a path P_1 from u to v and a path P_2 from v to w in a graph G then there is a path from u to w in G.

Proof: Composing P_1 and P_2 gives a walk from u to w.

$\exists$ a walk from u to w $\rightarrow$ $\exists$ a path from u to w.

Note: All of these facts are true for both directed and undirected graphs.