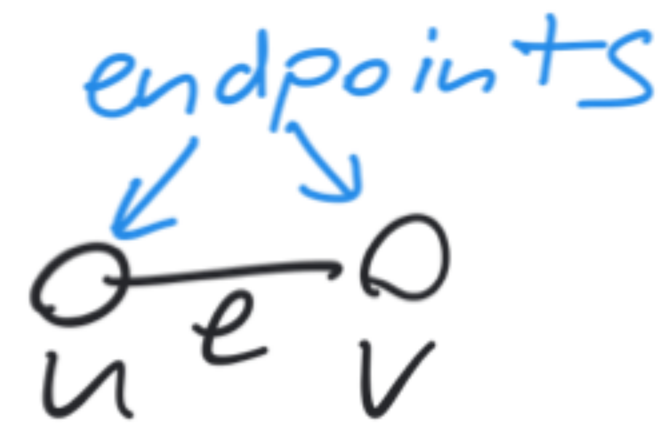
Vertex Degrees

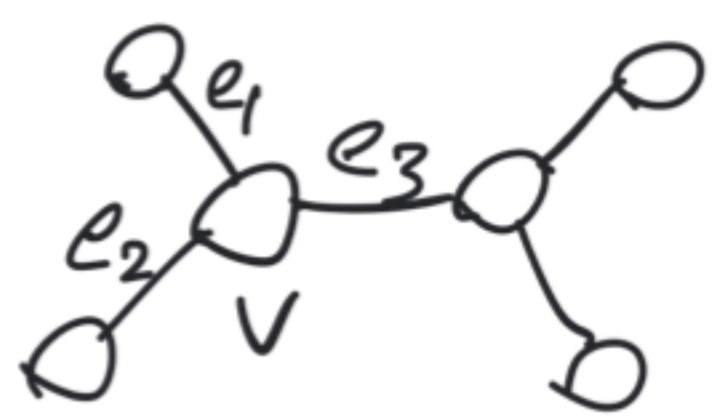
Definition: Given an edge $e = \{u, v\} \in E(G)$, we say that u and v are the **endpoints** of e.

Example:



Definition: We say that an edge e is **incident** to a vertex v if v is an endpoint of e.

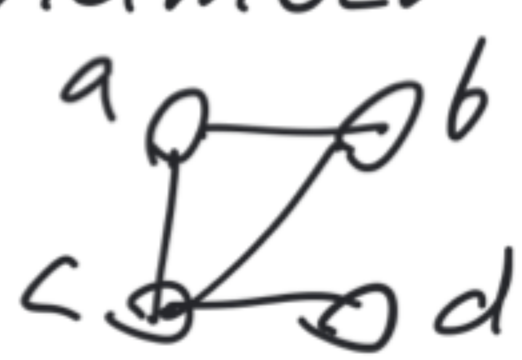
Example:



Edges e_1, e_2, e_3 are incident to v.

Definition: In an undirected graph G, the **degree** of a vertex v (which we write as $\deg(v)$) is the number of edges which are incident to v.

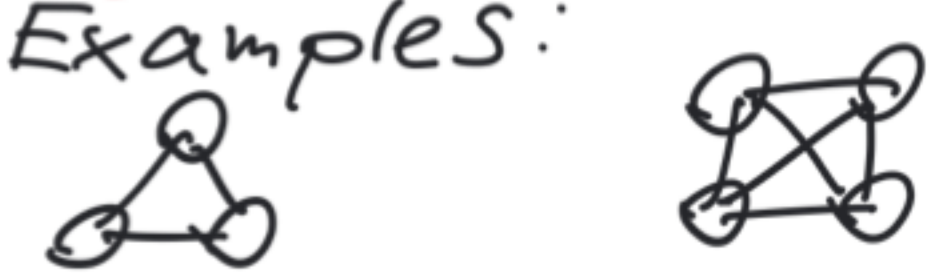
Example:



$\deg(a) = \deg(b) = 2$, $\deg(c) = 3$, and $\deg(d) = 1$.

Definition: An undirected graph G is **regular** if every vertex has the same degree.

Examples:



Sum of Vertex Degrees

Q: What is the sum of the degrees of the vertices of an undirected graph G?

Theorem: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

Idea: Every edge is incident to two vertices so it is counted twice.

Proof: Define 1_{ve} so that $1_{ve} = \begin{cases} 1 & \text{if } e \text{ is incident to } v \\ 0 & \text{otherwise} \end{cases}$

Consider $\sum_{v \in V(G)} \sum_{e \in E(G)} 1_{ve}$

$$\sum_{v \in V(G)} \sum_{e \in E(G)} 1_{ve} = \sum_{v \in V(G)} |\{e \in E(G) : e \text{ is incident to } v\}| = \sum_{v \in V(G)} \deg(v)$$

Switching the order of the summation,

$$\sum_{e \in E(G)} \sum_{v \in V(G)} 1_{ve} = \sum_{e \in E(G)} |\{v \in V(G) : v \text{ is an endpoint of } e\}| = \sum_{e \in E(G)} 2 = 2|E(G)|$$

Thus, $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.