Graph Theory Part 1

Discrete Mathematics 27100 Winter 2022

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1 Graphs

Definition 1.1. *A graph $G = (V(G), E(G))$ consists of a set of vertices $V(G)$ and a set of edges $E(G)$ where each edge $e \in E(G)$ is between two distinct vertices of $V(G)$.*

Graphs can be undirected or directed. For undirected graphs, the order of the vertices in each edge does not matter, i.e. $e = \{u, v\} = \{v, u\}$ is the edge between u and v. For directed graphs, the order of the vertices in each edge does matter, i.e. $e = (u, v)$ is the edge from u to v.

Note: Here we do not allow loops or multiple edges between a pair of vertices, though sometimes it is useful to allow this. When loops and/or multiple edges between a pair of vertices are allowed, we use the term multi-graph.

Remark 1.2. *We can think of the vertices as places and the edges as roads betwen the places.*

Definition 1.3. *Let G be an undirected graph.*

1. *We say that $u, v \in V(G)$ are adjacent if $\{u, v\} \in E(G)$*
2. *We say that a vertex v is incident to an edge e if $e = \{u, v\}$ for some $u \in V(G) \setminus \{v\}$. In other words, v is one of the two vertices of e.*
3. *We define the degree $deg(v)$ of a vertex $v \in V(G)$ to be $deg(v) = |\{e \in E(G) : v \text{ is incident to } e\}|$.*

2 Walks, Paths, and Cycles

Note: The following definitions and lemma apply for both undirected and directed graphs.

Definition 2.1. *A walk W from a vertex $s \in V(G)$ to a vertex $t \in V(G)$ is a sequence of edges $(v_0, v_1), (v_1, v_2), \dots, (v_{l-1}, v_l)$ in G where $v_0 = s$ and $v_l = t$. We define the length l of a walk W to be the number of edges it contains.*

Definition 2.2. *A path P from a vertex $s \in V(G)$ to a vertex $t \in V(G)$ is a sequence of edges $(v_0, v_1), (v_1, v_2), \dots, (v_{l-1}, v_l)$ in G where $v_0 = s$, $v_l = t$, and all of the vertices $v_0, v_1, \dots, v_l$ are distinct. In other words, a path is a walk which never visits the same vertex twice. We define the length l of a path P to be the number of edges it contains.*

Definition 2.3. *A cycle C is a sequence of edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{l-1}, v_l)$ where $v_v = v_0$ and all of the other vertices are distinct. In other words, a cycle is a walk which returns to its starting point at the end but otherwise has no repeated vertices. We define the length l of a cycle C to be the number of edges it contains.*

Lemma 2.4. *If G contains a walk from s to t then G then G contains a path from s to t.*

Proof. Let $W = (s, v_1), (v_1, v_2), \ldots, (v_{l-1}, t)$ be the shortest walk from s to t in G. We claim that W must be a path.

To prove this, we can use a proof by contradiction. Assume that $v_i = v_j$ for some $i < j$. If so, then let W' be the walk $W' = (s, v_1), \ldots (v_{i-1}, v_i), (v_j, v_{j+1}), \ldots, (v_{l-1}, t)$. Note that W' is also a walk from s to t in G and W' has length $i + (l - j) = l - (j - i) < l$. However, this contradicts our choice of W as the shortest walk from s to t in G. □

3 Connectivity and Cuts

Theorem 3.1. *There is a path from s to t in a graph G (directed or undirected) if and only if there is no cut $C = (L, R)$ such that*

1. *$L \cup R = V(G)$ and $L \cap R = \emptyset$ (i.e. L, R is a partition of $V(G)$)*
2. *$s \in L$ and $t \in R$*
3. *For all $u \in L$ and $v \in R$, $(u, v) \notin E(G)$*

Proof. If there is a path P from s to t then there cannot be such a cut C. To prove this, we can use a proof by contradiction.

Assume that there is both a path $P = (v_0, v_1), (v_1, v_2), \ldots, (v_{l-1}, v_l)$ from s to t and a cut $C = (L, R)$ such that

1. $L \cup R = V(G)$ and $L \cap R = \emptyset$
2. $s \in L$ and $t \in R$
3. For all $u \in L$ and $v \in R$, $\{u, v\} \notin E(G)$

Let v_j be the first vertex in P such that $v_j \in R$ (such a vertex must exist because $t \in R$). Note that $v_{j-1} \notin R$ as otherwise v_j would not be the first vertex in P which is in R. Thus, $v_{j-1} \in L$, $v_j \in R$, and $(v_{j-1}, v_j) \in E(G)$, which contradicts the third property of C.

If there is no path from s to t in G then we can construct $C = (L, R)$ as follows. Take $L = \{u : \text{ there is a path from } s \text{ to } u \text{ in G}\}$ and take $R = V(G) \setminus L$. We can see directly that C satisfies the first two properties. To see that C satisfies the third property, we can use a proof by contradiction.

Assume $\exists u \in L, v \in R\,((u, v) \in E(G))$. By definition, there is a path from s to u in G. Appending the edge (u, v) to this path, we obtain a walk from s to v so there must be a path from s to v in G. However, if this is the case then $v \in L$ so $v \notin R$. Contradiction. □

Definition 3.2. *We say that an undirected graph G is connected if for every pair of vertices $u, v \in V(G)$, there is a path from u to v in G.*

Corollary 3.3. *An undirected graph G is connected if and only if there does not exists a cut $C = (L, R)$ such that*

1. $L \cup R = V(G)$ *and* $L \cap R = \emptyset$
2. $L \neq \emptyset$ *and* $R \neq \emptyset$
3. $\forall u \in L, v \in R\,(\{u, v\} \notin E(G))$

Proof. If there exists a cut $C = (L, R)$ such that

1. $L \cup R = V(G)$ and $L \cap R = \emptyset$
2. $L \neq \emptyset$ and $R \neq \emptyset$
3. $\forall u \in L, v \in R\,(\{u, v\} \notin E(G))$

then by Theorem 3.1, for all $s \in L$ and all $t \in R$, there is no path from s to t in G so G is not connected. Conversely, if G is not connected then there exists $s, t \in V(G)$ such that there is no path from s to t in G. If so, by Theorem 3.1 there is a cut $C = (L, R)$ such that

1. $L \cup R = V(G)$ and $L \cap R = \emptyset$
2. $L \neq \emptyset$ and $R \neq \emptyset$
3. $\forall u \in L, v \in R\,((u, v) \notin E(G))$

□

4 Eulerian Circuits and Eulerian Trails

Definition 4.1. *An Eulerian trail on an undirected graph G is a walk W which uses each edge of G exactly once. An Euleriean circuit is an Eulerian tour which starts and ends at the same vertex.*

The key property of a connected G which determines whether or not it has Eulerian circuits and whether it has Eulerian trails is which of its vertices have odd degree.

We first observe that for an undirected graph G, the sum of its degrees is even.

Theorem 4.2. *For all undirected graphs G, $\sum_{v \in V(G)} deg(v) = 2|E(G)|$*

Proof sketch. For each vertex $v \in V(G)$ and each edge $e \in E(G)$, let $x_{ve} = 1$ if v is incident to e (i.e. v is one of the two vertices of e) and let $x_{ve} = 0$ otherwise. Now observe that

$$\sum_{v \in V(G)} deg(v) = \sum_{v \in V(G)} \sum_{e \in E(G)} x_{ve} = \sum_{e \in E(G)} \sum_{v \in V(G)} x_{ve} = \sum_{e \in E(G)} 2 = 2|E(G)|$$

□

We now describe when connected undirected graphs have Eulerian circuits and Eulerian trails.

Theorem 4.3. *Let G be a connected undirected graph.*

1. *For all vertices $s \in V(G)$, G has an Eulerian circuit starting and ending at s if and only if* $\forall v \in V(G), deg(v) \equiv 0 \mod 2$

2. *For all pairs of distinct vertices $s, t \in V(G)$, G has an Eulerian trail starting at s and ending at t if and only if $deg(s) \equiv deg(t) \equiv 1 \mod 2$ and $\forall v \in V(G) \setminus \{s, t\}$, $deg(v) \equiv 0 \mod 2$.*

Proof. We first make the following key observation:

Definition 4.4. *Given a walk W on an undirected graph G, define G_W to be the graph where*

1. $V(G_W) = \{v \in V(G) : v$ *is incident to an edge in* $W\}$

2. $E(G_W) = E(W)$ *where $E(W)$ is the set of edges in W.*

Lemma 4.5. *If W is a walk on an undirected graph G which uses each edge at most once then*

1. *If W starts and ends at some vertex s then every vertex in G_W has even degree.*

2. *If W starts at a vertex s and ends at a different vertex t then in G_W, s and t have odd degree and every other vertex of G_W has even degree.*

Proof. Consider the number of times that W enters and leaves each vertex. If v is a vertex in $V(W)$ which is not an endpoint then each time W enters v, W leaves v immediately afterwards. Thus, the number of times W enters v is equal to the number of times W leaves v and $deg(v)$ is even.

If W both starts and ends at s then the number of times W enters s is equal to the number of times W leaves s because the time when W leaves s at the beginning is balanced by the time when W enters t at the end.

If W starts at a vertex s and ends at a different vertex t then W leaves s one more time than W enters s (as W leaves s at the beginning) and W enters t one more time than W leaves t (as W enters t at the end). Thus, $deg(s)$ and $deg(t)$ are odd in G_W □

Now observe that if W is an Eulerian trail or an Eulerian circuit then $G_W = G$. Thus, if G has an Eulerien circuit starting and ending at s then all vertices of G have even degree and if G has an Eulerian trail starting at s and ending at t then s and t have odd degree and all other vertices of G have even degree.

To show that this condition is sufficient, let W be the longest walk in G from s to itself (if all vertices of G have even degree) or from s to t (if s and t are the only vertices in G with odd degree) which uses each edge at most once. We claim that W must contain all of the edges of G. To see this, assume that W does not contain all of the edges of G and let $E' = E(G) \setminus E(W)$. Observe that in the graph $G' = (V(G), E')$, every vertex has even degree.

There must be a vertex s' reached by W which is incident to an edge E'. Now take a walk W' in G' where we start at s' and continue as long as there is a new edge to use. Whenever we are not at s', there must be an odd number of edges left to take so we can only stop at s'. This means that we can make W longer by inserting W' when we first reach s'. This contradicts the assumption that W is the longest length walk in G from s to itself (if all vertices of G have even degree) or from s to t (if s and t are the only vertices in G with odd degree) which uses each edge at most once. □