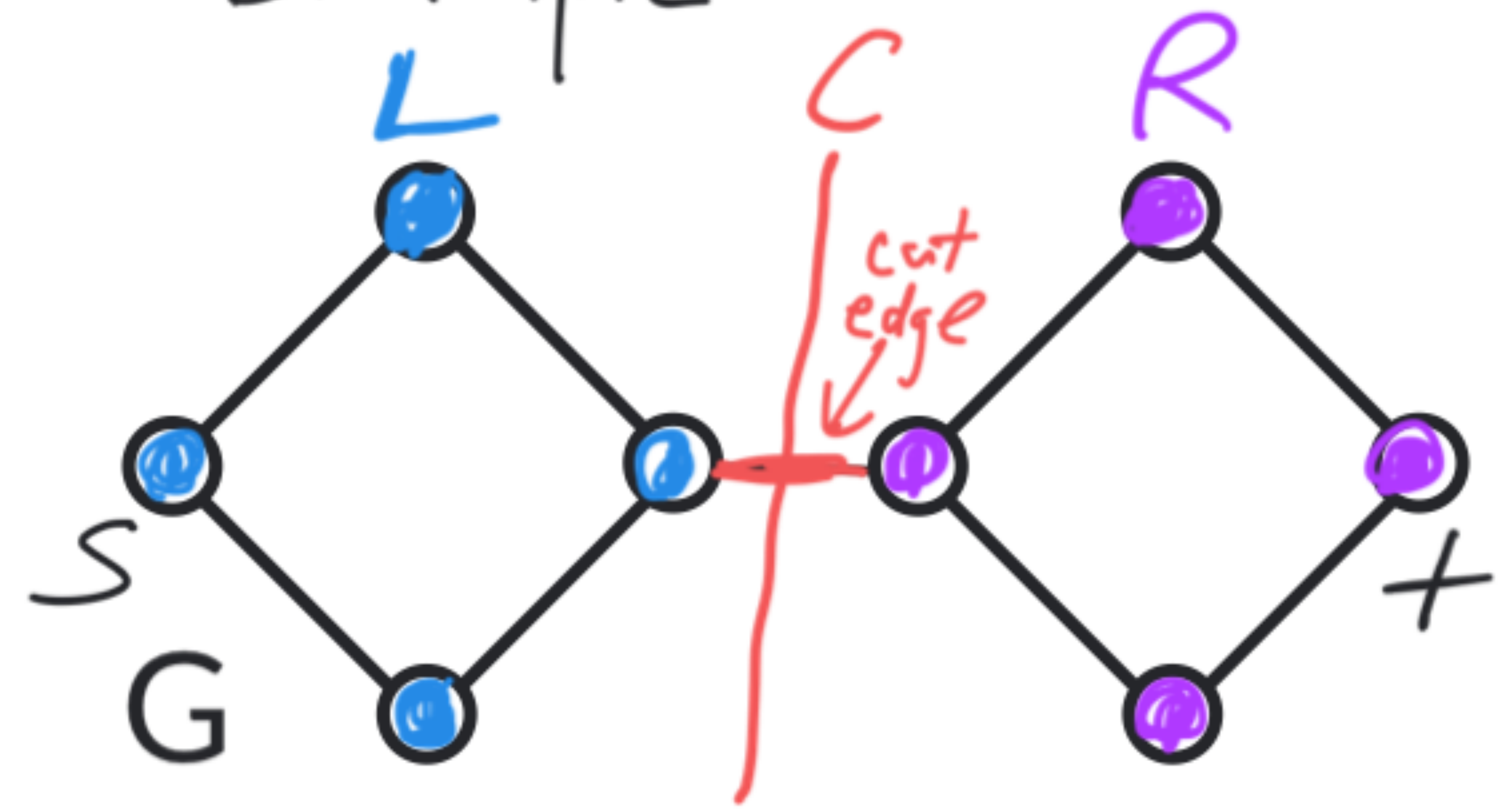
Cuts and Connectivity

Definition: Given a graph G, a cut $C=(L,R)$ is a partition of $V(G)$ into two parts L,R.

Example:



We say that C cuts an edge $e=\{u,v\}$ if u,v are on opposite sides of C.

Theorem: Given a graph G and $s,t \in V(G)$, there is no path from s to t in G if and only if there is a cut $C=(L,R)$ such that $s \in L$, $t \in R$, and there is no edge $e=\{u,v\} \in E(G)$ such that $u \in L$ and $v \in R$ (i.e. C does not cut any edges).

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Theorem: Given a graph G and two vertices s,t of G, there is no path from s to t in G if and only if there is a cut C=(L,R) such that s is in L, t is in R, and there is no edge e={u,v} of G such that u is in L and v is in R (i.e. C does not cut any edges of G).

Proof:

1. Cut $C \rightarrow$ there is no path from s to t

Assume we have such a cut $C=(L,R)$ and a path $P=\{v_0,v_1\},\{v_1,v_2\},\dots,\{v_{\ell-1},v_\ell\}$ where $v_0=s$ and $v_\ell=t$.
Let j be the first index such that $v_j \in R$. $v_{j-1} \in L$ so C cuts the edge $\{v_{j-1},v_j\}$. Contradiction.

2. There is no path from s to t $\rightarrow \exists$ cut C.

Let $L=\{v \in V(G):$ there is a path from s to v in $G\}$ and let $R=V(G)\setminus L$.
$s \in L$ and $t \in R$. Assume there is an edge $e=\{u,v\} \in E(G)$ such that $u \in L$ and $v \in R$. Since $u \in L$, there is a path from s to u. Since $\{u,v\} \in E(G)$ there is a path from s to v so $v \in L$. Contradiction.

Note: This proof also works for directed graphs. Condition on C: C does not cut any edge $e=(u,v)$ from L to R.