Solving Inhomogeneous Recurrence Relations

Q: How do we solve inhomogeneous recurrence relations such as $\underline{b_n = 2b_{n-1}}\ \underline{-1}$?

Idea: Split the recurrence relation into a homogeneous part and an inhomogeneous part.

IF the inhomogeneous part is <u>constant</u> or <u>only depends on n</u>, we can proceed as follows:

1. Find the general solution to the homogeneous part.
2. Find a single solution to the <u>entire recurrence relation</u>.
3. Add the general solution to the homogeneous part to our single solution to the entire recurrence relation.

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Example: $b_n = 2b_{n-1} - 1$

Homogeneous part: $b_n = 2b_{n-1}$

Trying $b_n = x^n$, $x^n = 2x^{n-1}$.

Dividing both sides by x^{n-1}, $x = 2$.

General solution: $b_n = c_1 \cdot 2^n$

To obtain a solution to the entire recurrence relation, let's try $c \cdot$(inhomogeneous part) for some constant c as an educated guess.

Trying $b_n = c$, $c = 2c - 1$ so $c = 1$.

One solution: $b_n = 1$.

General solution: $b_n = c_1 \cdot 2^n + 1$.

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Example: $b_n = 2b_{n-1} - 1$

Q: Why do we take $b_n = \underbrace{f(n)}_{\text{arbitrary solution to the homogeneous part}} + \underbrace{g(n)}_{\text{one solution to the entire recurrence relation}}$?

Assume $g(n)$ is a solution and consider when $f(n)+g(n)$ is a solution.

Since $g(n) = 2g(n-1) - 1$,

$$f(n) + g(n) = 2(f(n-1) + g(n-1)) - 1$$

$$\Updownarrow$$

$$f(n) + 2g(n-1) - 1 = 2f(n-1) + 2g(n-1) - 1$$

$$\Updownarrow$$

$$f(n) = 2f(n-1)$$

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The first few number(s) will determine the coefficients in the solution.

Example: $b_1=2,\ b_n=2b_{n-1}-1$

General solution: $b_n = c_1 \cdot 2^n + 1$

$b_1 = 2$ so $2 = 2c_1 + 1$ and thus $c_1 = \frac{1}{2}$.

Solution: $b_n = \frac{1}{2} \cdot 2^n + 1 = 2^{n-1} + 1$