Recurrence Relations

Discrete Mathematics Winter 2022

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1 Introduction

Many sequences are defined by their first few elements and a recurrence relation which describes how to obtain a given element of the sequence from the previous elements of the sequence.

Example 1.1. *The Fibonacci numbers are defined as follows:*

1. $F_1 = 1$
2. $F_2 = 1$
3. *For all natural numbers* $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$

The first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

In this lecture, we describe how to find closed form expressions for many sequences which are defined by recurrence relations.

2 Homogeneous Recurrence Relations

For simplicity, we start with homogeneous recurrence relations.

Definition 2.1. *A recurrence relation is homogeneous if it only involves terms of the form* ca_m *for some where* c *is a constant and* m *depends on* n *(we will generally have* $m = n + c'$ *for some constant* c'*).*

Example 2.2. *Some examples of homogeneous and inhomogenous recurrence relations are as follows:*

1. *The recurrence relation* $F_n = F_{n-1} + F_{n-2}$ *for the Fibonacci numbers is homogeneous*
2. *The recurrence relation* $a_n = 2a_{n-1} + 1$ *is not homogeneous because of the* $+1$ *term.*
3. *The recurrence relation* $a_n = a_{n-1}^2$ *is not homogeneous because* a_{n-1} *is squared.*

Lemma 2.3. *Given a homogeneous recurrence relation, for any solutions $f(n)$ and $g(n)$ of the recurrence relation and any real numbers r and s, $rf + sg$ is also a solution to the recurrence relation.*

Proof. A homogeneous recurrence relation has the form

$$\text{For all } n \geq n_0, a_n = \sum_{i=1}^{k} c_i a_{m_i}$$

where each m_i is a function of n. If f and g are solutions to the recurrence relation then

1. For all $n \geq n_0$, $f(n) = \sum_{i=1}^{k} c_i f(m_i)$
2. For all $n \geq n_0$, $g(n) = \sum_{i=1}^{k} c_i g(m_i)$

If so, for all real numbers r and s,

$$\text{For all } n \geq n_0, rf(n) + sg(n) = r\sum_{i=1}^{k} c_i f(m_i) + s\sum_{i=1}^{k} c_i g(m_i) = \sum_{i=1}^{k} c_i \left(rf(m_i) + sg(m_i)\right)$$

□

2.1 Educated guessing and checking for Homogeneous Recurrence Relations

One way to solve recurrence relations is by taking educated guesses, checking them, and adjusting accordingly. In particular, for homogeneous recurrence relations where every term is of the form $ca_{n+c'}$ where c is a real number and c' is an integer, there will be solutions of the form x^n for one or more $x \in \mathbb{R} \setminus \{0\}$ ($x = 0$ gives a solution as well but this is trivial). By plugging in $a_n = x^n$, we can find these solution(s).

If we are also given initial value(s) for the recurrence relation, once the solutions $x_1, \ldots, x_k$ for x have been found we can plug in $a_n = \sum_{i=1} c_i x_i^n$ to find the coefficient c_i for each term x_i^n.

Example 2.4. *For the Fibonacci numbers, the recurrence relation is $F_n = F_{n-1} + F_{n-2}$. Plugging in $F_n = x^n$ gives $x^n = x^{n-1} + x^{n-2}$. Dividing this by x^{n-2} and rearranging gives $x^2 - x - 1 = 0$. Solving this for x gives $x = \frac{1\pm\sqrt{5}}{2}$.*

Since $F_1 = 1$ and $F_2 = 1$, plugging in $F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ we have that

1. $\left(\frac{1+\sqrt{5}}{2}\right)c_1 + \left(\frac{1-\sqrt{5}}{2}\right)c_2 = 1$
2. $\left(\frac{3+\sqrt{5}}{2}\right)c_1 + \left(\frac{3-\sqrt{5}}{2}\right)c_2 = 1$

Solving these equations gives $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$. Thus, the Fibonacci numbers have the following closed form expression:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example 2.5. *Consider the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ where $a_1 = 1$ and $a_2 = 13$. Plugging in $a_n = x^n$ gives $x^n = x^{n-1} + 5x$. Dividing this by x^{n-2} and rearranging gives $x^2 - x - 6 = 0$. Solving this for x gives $x = 3$ or $x = -2$.*

Since $a_1 = 1$ and $a_2 = 13$, plugging in $a_n = c_1(3^n) + c_2(-2)^n$ we have that

1. $3c_1 - 2c_2 = 1$
2. $9c_1 + 4c_2 = 13$

Solving these equations gives $c_1 = 1$ and $c_2 = 1$. Thus, this sequence has the following closed form expression:

$$a_n = 3^n + (-2)^n$$

Remark 2.6. *If the polynomial for x has a double root then the general solution to the recurrence relation is somewhat more complicated.*

3 Inhomogeneous Recurrence Relations

Often we will need to handle inhomogeneous recurrence relations. We now describe how to handle recurrence relations of the form $a_n = \sum_{i=1}^{k} c_i a_{m_i} + h(n)$ where we have a homogeneous recurrence relations plus an additional inhomogeneous term $h(n)$.

Lemma 3.1. *If $f(n)$ and $g(n)$ are two solutions to the recurrence relation $a_n = \sum_{i=1}^{k} c_i a_{m_i} + h(n)$ then $f(n) - g(n)$ is a solution to the recurrence relation $a_n = \sum_{i=1}^{k} c_i a_{m_i}$*

Proof. If f and g are solutions to the recurrence relation $a_n = \sum_{i=1}^{k} c_i a_{m_i} + h(n)$ then

1. For all $n \geq n_0$, $f(n) = \sum_{i=1}^{k} c_i f(m_i) + h(n)$
2. For all $n \geq n_0$, $g(n) = \sum_{i=1}^{k} c_i g(m_i) + h(n)$

If so, for all $n \geq n_0$,

$$f(n) - g(n) = \sum_{i=1}^{k} c_i f(m_i) + h(n) - \sum_{i=1}^{k} c_i g(m_i) - h(n) = \sum_{i=1}^{k} c_i \left(f(m_i) - g(m_i)\right)$$

□

Based on this, we have the following strategy for solving recurrence relations of the form $a_n = \sum_{i=1}^{k} c_i a_{m_i} + h(n)$:

1. Find the general solution $f(n)$ for the homogeneous part of the recurrence relation $a_n = \sum_{i=1}^{k} c_i a_{m_i} + h(n)$.
2. Find a single solution $g(n)$ for the entire recurrence relation.
3. Add the two parts together to obtain the general solution $f(n) + g(n)$ to the recurrence relation.

3.1 Educated guessing and checking for Inhomogeneous Recurrence Relations

We can still solve inhomogeneous recurrence relations by taking educated guesses, checking them, and adjusting accordingly. The homogeneous part of the recurrence relation can be solved in the same way as before.

For the inhomogeneous part of the solution, when $h(n) = c$ for some constant c or $h(n) = cr^n$ for some constants c and r, taking $g(n) = c'h(n)$ for another constant c' usually works.

Example 3.2. *Consider the recurrence relation $a_n = 2a_{n-1} + 1$ where $a_1 = 1$. For the homogeneous part $a_n = 2a_n$ of the recurrence relation, plugging in x^n gives $x^n = 2x^{n-1}$. Dividing by x^{n-1} gives $x = 2$. Thus, the solutions to the homogeneous part of the recurrence relation are functions of the form $f(n) = c2^n$*

For the inhomogeneous part of the equation, we can try $a_n = c'$. Plugging in $a_n = c'$ gives $c' = 2c' + 1$. Solving this gives $c' = -1$ so $g(n) = -1$ is a solution to the entire recurrence relation. Putting these pieces together, $a_n = c2^n - 1$ is the general solution to this recurrence relation.

When $a_1 = 1$, we have that $1 = 2c - 1$ so $c = 1$. Thus, the solution when $a_1 = 1$ is $a_n = 2^n - 1$.

Remark 3.3. *If plugging in $g(n) = ch(n)$ fails because the homogeneous part of the equation becomes 0, trying $g(n) = c'nh(n)$ often works. Also, if $h(n) = n^p$ then $g(n) = ch(n)$ doesn't quite work but does make progress by canceling out the n^p term and leaving lower degree terms.*

3.2 Expanding Out Recurrence Relations

When the recurrence relations are relatively simple, they can also be solved by recursively plugging the recurrence relations into themselves. This often results in geometric series, so we first recall how to evaluate geometric series here.

Lemma 3.4. $\sum_{j=0}^{k} r^k = \frac{r^{k+1}-1}{r-1}$

Proof. Let $x = \sum_{j=0}^{k} r^k$. Observe that

1. $x = 1 + r + \ldots + r^k$
2. $rx = r + \ldots + r^k + r^{k+1}$

Thus, $rx - x = r^{k+1} - 1$ and dividing both sides by $r - 1$ gives $x = \frac{r^{k+1}-1}{r-1}$. □

Example 3.5. *Consider the recurrence relation $a_n = 2a_{n-1} + 1$. Recursively plugging this recurrence relation into itself we have that:*

1. $a_n = 2a_{n-1} + 1$
2. $a_n = 4a_{n-2} + 2 + 1$
3. $a_n = 8a_{n-3} + 4 + 2 + 1$
4. $a_n = 2^k a_{n-k} + \sum_{j=0}^{k-1} 2^j = 2^k a_{n-k} + \frac{2^k-1}{2-1} = 2^k a_{n-k} + 2^k - 1$

Plugging in $k = n - 1$ we have that $a_n = 2^{n-1}a_1 + 2^{n-1} - 1$. When $a_1 = 1$ this gives that $a_n = 2^n - 1$.