Homogeneous Recurrence Relations

Homogeneous: Every term has the same degree (as a polynomial in $a_1, a_2, a_3, \dots$)

Examples	Homogeneous?
Fibonacci numbers: $F_n = F_{n-1} + F_{n-2}$	✓
$a_n = a_{n-1} + 2$ Degree 0	✗
$b_n = b_{n-1}^2$ Degree 2	✗

Homogeneous Recurrence Relations

Claim: For any homogeneous recurrence relation, the set of solutions forms a vector space.

In other words, if $f(n)$ and $g(n)$ are two solutions, then for any constants c_1 and c_2, $c_1 f + c_2 g$ is also a solution.

Example: $a_n = 3a_{n-1} - 2a_{n-2}$

If $f(n) = 1$, then $a_n = f(n) = 1$ is a solution as $1 = 3 - 2$.

If $g(n) = 2^n$, then $a_n = g(n) = 2^n$ is a solution as

$$3 \cdot 2^{n-1} - 2 \cdot 2^{n-2} = \left(\frac{3}{2} - \frac{2}{4}\right) 2^n = 2^n$$

Now for any constants c_1 and c_2, $a_n = c_1 + c_2 \cdot 2^n$ is a solution as

$$3c_1 + 3c_2 \cdot 2^{n-1} - 2c_1 - 2c_2 \cdot 2^{n-2} = (3-2)c_1 + \left(\frac{3}{2} - \frac{2}{4}\right) \cdot c_2 2^n = c_1 + c_2 \cdot 2^n$$

Homogeneous Recurrence Relations

Q: How can we solve these recurrence relations?

Idea: Try x^n as an educated guess.

This works well because most x terms will cancel, giving us a polynomial equation which we can solve for x.

Examples:

If $b_n = 3b_{n-1}$ then trying $b_n = x^n$ gives $x^n = 3x^{n-1}$.
Dividing both sides by x^{n-1} gives $x = 3$.
General solution: $b_n = c_1 \cdot 3^n$.

If $a_n = 3a_{n-1} - 2a_{n-2}$ then trying $a_n = x^n$ gives $x^n = 3x^{n-1} - 2x^{n-2}$
Dividing both sides by x^{n-2} gives $x^2 = 3x - 2$ so
$x^2 - 3x + 2 = 0$. $x = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$. x is 1 or 2.
General solution: $a_n = c_1 \cdot 1^n + c_2 \cdot 2^n$

Homogeneous Recurrence Relations

Q: How can we determine the coefficients for the solution?

A: The first few numbers in the sequence will determine these coefficients.

Example:

If $b_1 = 2$ and $b_n = 3b_{n-1}$,

General solution: $b_n = c_1 \cdot 3^n$

$b_1 = 2$ so $2 = 3c_1$ and thus $c_1 = \frac{2}{3}$.

Solution: $b_n = \frac{2}{3} \cdot 3^n = 2 \cdot 3^{n-1}$