Expanding Out Recurrence Relations

Example 1: $a_1=1,\ a_n=2a_{n-1}+3$

Geometric series formula: $\sum_{j=0}^{k} r^j = 1+r+r^2+\ldots+r^k = \frac{r^{k+1}-1}{r-1}$

$a_{n-1} = 2a_{n-2}+3$

$a_{n-2} = 2a_{n-3}+3$

Step 1: $a_n = 2a_{n-1}+3$

2: $a_n = 2(2a_{n-2}+3)+3$

$= 4a_{n-2}+2\cdot 3+3$

3: $a_n = 4(2a_{n-3}+3)+2\cdot 3+3$

$= 8a_{n-3}+4\cdot 3+2\cdot 3+3$

Idea: See what we have after k steps.

After k steps: $a_n = 2^k a_{n-k} + 3(1+2+4+\ldots+2^{k-1})$

$a_n = 2^k a_{n-k} + 3\left(\sum_{j=0}^{k-1} 2^j\right)$

$\sum_{j=0}^{k-1} 2^j = \frac{2^k-1}{2-1} = 2^k-1$

$a_n = 2^k a_{n-k} + 3(2^k-1)$

$= (3+a_{n-k})2^k - 3$

Now we plug in $k=n-1$ to put this in terms of a_1

$a_n = (3+a_1)2^{n-1} - 3$

Expanding Out Recurrence Relations

Example 1: $a_1=1,\ a_n=2a_{n-1}+3$

General solution: $a_n=(3+a_1)2^{n-1}-3$

Specific solution: $a_n=(3+1)2^{n-1}-3$

$=4\cdot 2^{n-1}-3$

$=2^{n+1}-3$

Checking our answer:

n	1	2	3	4	5
a_n	1	5	13	29	61
$2^{n+1}-3$	1	5	13	29	61

✓

Expanding Out Recurrence Relations

Example 2: $b_1 = 1,\ b_n = 3b_{n-1} - 1$

Geometric series formula: $\sum_{j=0}^{k} r^j = 1 + r + r^2 + \ldots + r^k = \frac{r^{k+1} - 1}{r - 1}$

Step 1: $b_n = 3b_{n-1} - 1$

$b_{n-1} = 3b_{n-2} - 1$

2: $b_n = 3(3b_{n-2} - 1) - 1$

$= 9b_{n-2} - 3 - 1$

$b_{n-2} = 3b_{n-3} - 1$

3: $b_n = 9(3b_{n-3} - 1) - 3 - 1$

$= 27b_{n-3} - 9 - 3 - 1$

After k steps: $b_n = 3^k b_{n-k} - (1 + 3 + 9 + \ldots + 3^{k-1})$

$= 3^k b_{n-k} - \sum_{j=0}^{k-1} 3^j$

$\sum_{j=0}^{k-1} 3^j = \frac{3^k - 1}{3 - 1} = \frac{3^k - 1}{2}$

$b_n = 3^k b_{n-k} - \left(\frac{3^k - 1}{2}\right)$

$= \left(b_{n-k} - \frac{1}{2}\right)3^k + \frac{1}{2}$

Plugging in $k = n-1$, $\boxed{b_n = \left(b_1 - \frac{1}{2}\right)3^{n-1} + \frac{1}{2}}$

Expanding Out Recurrence Relations

Example 2: $b_1 = 1,\ b_n = 3b_{n-1} - 1$

General solution: $b_n = (b_1 - \frac{1}{2})3^{n-1} + \frac{1}{2}$

Plugging in $b_1 = 1$, $b_n = (1 - \frac{1}{2})3^{n-1} + \frac{1}{2}$

$$b_n = \frac{3^{n-1} + 1}{2}$$

Checking our answer:

n	1	2	3	4	5
b_n	1	2	5	14	41
$\frac{3^{n-1}+1}{2}$	1	2	5	14	41

✓

Expanding Out Recurrence Relations

Important note:

Expanding out the recurrence relation works well for recurrence relations like $a_n = 2a_{n-1} + 3$ where the right hand side has one term of the sequence.
Expanding out the recurrence relation does not work so well for recurrence relations like the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$ where the right hand side has multiple terms of the sequence.