Variance

Definition: Given a probability space $(\Omega, \Pr)$ and a random variable $X: \Omega \to \mathbb{R}$, the variance of X is $Var(X) = \sum_{\omega \in \Omega} \Pr(\omega)\left(X(\omega) - E[X]\right)^2 = E\left[(X - E[X])^2\right]$

Proposition: $Var(X) = \sum_{x \in Range(X)} \Pr(X=x)\,(x - E[X])^2$

Example 1: If we flip a fair coin 3 times and $X = \#$ of heads, what is $Var(X)$?

x	0	1	2	3
$\Pr(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$(x-E[X])^2$	$\frac{9}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{9}{4}$

$E[X] = \frac{3}{2}$

$Var(X) = 2 \cdot \frac{1}{8} \cdot \frac{9}{4} + 2 \cdot \frac{3}{8} \cdot \frac{1}{4} = \frac{9+3}{16} = \frac{12}{16} = \frac{3}{4}$

Example 2: If we roll a fair 6-sided die and Y is the result, what is $Var(Y)$?

y	1	2	3	4	5	6
$\Pr(Y=y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$(y-E[Y])^2$	$\frac{25}{4}$	$\frac{9}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{25}{4}$

$E[Y] = \frac{7}{2}$

$Var(Y) = \frac{1}{6}\left(\frac{25}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{25}{4}\right)$

$= \frac{70}{24} = \frac{35}{12}$

Notes on Variance

$$\mathrm{Var}(X) = E\left[\left(X - E[X]\right)^2\right]$$

Definition: Given a random variable X the standard deviation of X is $\sigma = \sqrt{\mathrm{Var}(X)}$.

Observation: $\sigma = \mathrm{Var}(X) = 0 \iff X$ is constant

Theorem: $\mathrm{Var}(X) = E[X^2] - (E[X])^2$

Proof:

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X - E[X])^2] \\
&= E[X^2 - 2E[X]X + (E[X])^2] \\
&= E[X^2] - 2E[X]E[X] + (E[X])^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}$$

Variance of the Sum of Independent Random Variables

Definition: Given a probability space (Ω, Pr) and two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$, we say that X and Y are independent if

$$\forall x \in Range(X), \forall y \in Range(Y), \; Pr(X=x, Y=y) = \sum_{\omega \in \Omega : X(\omega)=x, Y(\omega)=y} Pr(\omega) = Pr(X=x)Pr(Y=y)$$

Lemma: If X and Y are <u>independent</u> random variables then $E[XY]=E[X]E[Y]$

Proof: $E[XY] = \sum_x \sum_y Pr(X=x, Y=y)xy = \sum_x \sum_y Pr(X=x)x \, Pr(Y=y)y$

$$= \sum_x Pr(X=x)x \sum_y Pr(Y=y)y = E[X]E[Y]$$

Theorem: If X and Y are <u>independent</u> random variables then $Var(X+Y) = Var(X) + Var(Y)$

Proof: $Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$

$$= E[X^2] + 2E[XY] + E[Y^2] - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2$$

$$= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2 + 2E[XY] - 2E[X]E[Y]$$

$$= Var(X) + Var(Y)$$

Variance of the Sum of Independent Random Variables

Theorem: If X and Y are independent random variables then
Var(X+Y)=Var(X)+Var(Y)

Example: If we flip a fair coin k times and $X = \#$ of heads, what is $Var(X)$?

Trick: Write $X = \sum_{i=1}^{k} X_i$ where $X_i = \begin{cases} 1 & \text{if the } i\text{th coin is heads} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = \frac{1}{2} \qquad E[X_i^2] = E[X_i] = \frac{1}{2}$$

$$Var(X_i) = E[X_i^2] - (E[X_i])^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$Var(X) = \sum_{i=1}^{k} Var(X_i) = \sum_{i=1}^{k} \frac{1}{4} = \frac{k}{4}$$