Random Variables

Intuition: A random variable is a variable whose value depends on what happens.

Definition: Given a probability space $(\Omega, \Pr)$, a random variable is a function $X: \Omega \to \mathbb{R}$.

Example: Let's say we flip a fair coin three times.

X = # of heads is a random variable.

$$X(TTT) = 0$$

$$X(HTT) = X(THT) = X(TTH) = 1$$

$$X(HHT) = X(HTH) = X(THH) = 2$$

$$X(HHH) = 3$$

Probability Spaces for Random Variables

For a random variable X, it's useful to think about which values X can take and the probability that X takes each possible value x.

Proposition: Given a probability space $(\Omega, \Pr)$ and given a random variable $X: \Omega \to \mathbb{R}$, if we take $\mathrm{Range}(X) = \{X(\omega) : \omega \in \Omega\}$ to be the sample space and take the probability distribution $\Pr(X = x) = \sum_{\omega \in \Omega : X(\omega) = x} \Pr(\omega)$ then this is a probability space.

Example: If we flip a fair coin 3 times and $X = \#$ of heads then

$$\mathrm{Range}(X) = \{0, 1, 2, 3\}$$

$$\Pr(X=0) = \frac{1}{8}, \quad \Pr(X=1) = \frac{3}{8}, \quad \Pr(X=2) = \frac{3}{8}, \quad \Pr(X=3) = \frac{1}{8}.$$