Expected Values

Definition: Given a probability space $(\Omega, \Pr)$ and a random variable $X : \Omega \to \mathbb{R}$, the expected value of X is $E[X] = \sum_{w \in \Omega} \Pr(w) \cdot X(w)$

Proposition: $E[X] = \sum_{x \in Range(X)} \Pr(X=x)x$

Proof: $E[X] = \sum_{w \in \Omega} \Pr(w) X(w) = \sum_{x \in Range(X)} \sum_{w \in \Omega : X(w)=x} \Pr(w) X(w)$

$= \sum_{x \in Range(X)} x \left(\sum_{w \in \Omega : X(w)=x} \Pr(w) \right) = \sum_{x \in Range(X)} \Pr(X=x) \cdot x$

Example 1: If we flip a fair coin 3 times and X is the # of heads then $E[X] = \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3 = \frac{3+6+3}{8}$

$= \frac{12}{8} = \frac{3}{2}$

Example 2: If Y is the result of rolling a fair 6-sided die then $E[Y] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{1+2+3+4+5+6}{6}$

$= \frac{21}{6} = \frac{7}{2} = 3\frac{1}{2}$

Linearity of Expectation

Linearity of Expectation: Given a probability space (Ω, Pr) and random variables $X: \Omega \to \mathbb{R}$ and $Y: \Omega \to \mathbb{R}$, $E[X+Y] = E[X] + E[Y]$

Proof: $E[X+Y] = \sum_{\omega \in \Omega} Pr(\omega)(X(\omega)+Y(\omega)) = \left(\sum_{\omega\in\Omega} Pr(\omega)X(\omega)\right) + \left(\sum_{\omega \in \Omega} Pr(\omega)Y(\omega)\right) = E[X]+E[Y]$

Example: If we flip a fair coin k times and $X = \#$ of heads, what is $E[X]$?

Trick: Write $X = \sum_{i=1}^{k} X_i$ where $X_i = 1$ if the ith coin is heads, 0 otherwise

$$E[X_i] = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$E[X] = \sum_{i=1}^{k} E[X_i] = \sum_{i=1}^{k} \frac{1}{2} = \frac{k}{2}$$