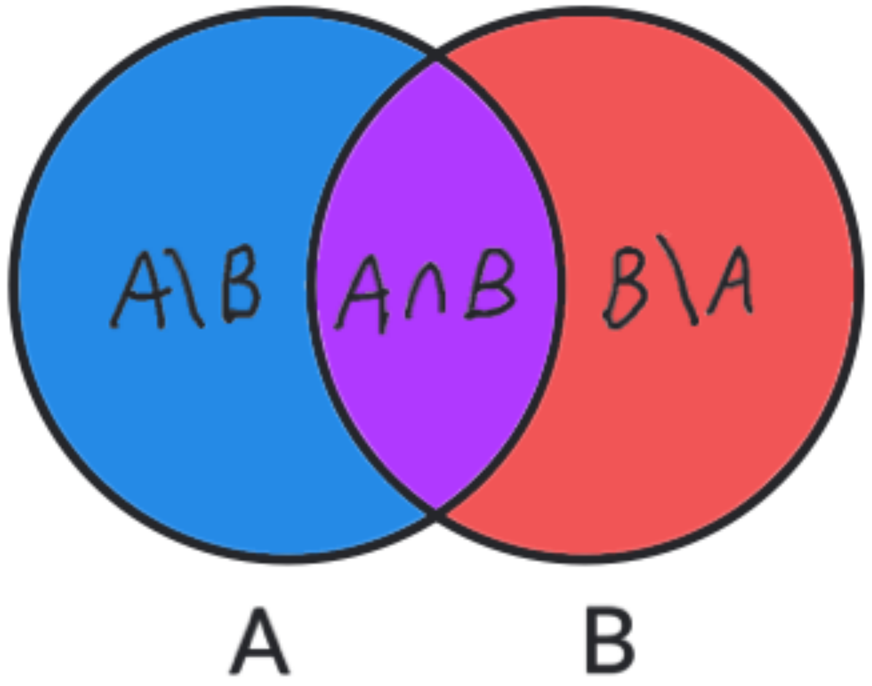
# Inclusion-Exclusion Principle




Inclusion-Exclusion Principle:

Combinatorics: $|A \cup B| = |A| + |B| - |A \cap B|$

Probability Theory: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

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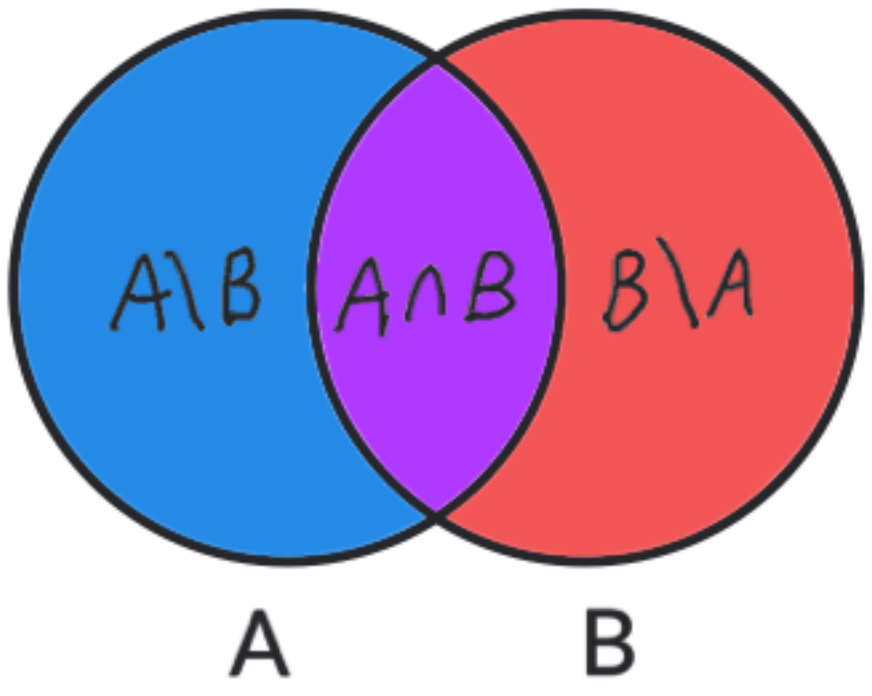
Example: If we roll 2 fair 6-sided dice, what is the probability that at least one of them is a 6?

A: The first die is a 6 B: The second die is a 6

$Pr(A) = \frac{1}{6}$, $Pr(B) = \frac{1}{6}$, and $Pr(A \cap B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

$Pr(A \cup B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{2}{6} - \frac{1}{36} = \frac{12}{36} - \frac{1}{36} = \frac{11}{36}$

# Inclusion-Exclusion Principle Proof




Proof that $|A \cup B| = |A| + |B| - |A \cap B|$:

Observe that $|A| = |A \setminus B| + |A \cap B|$ and $|B| = |B \setminus A| + |A \cap B|$.

$|A| + |B| - |A \cap B| = |A \setminus B| + |B \setminus A| + |A \cap B| = |A \cup B|$

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Proof that $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$:

$$Pr(A) = \sum_{w \in A} Pr(w) = \sum_{w \in A \setminus B} Pr(w) + \sum_{w \in A \cap B} Pr(w) = Pr(A \setminus B) + Pr(A \cap B)$$

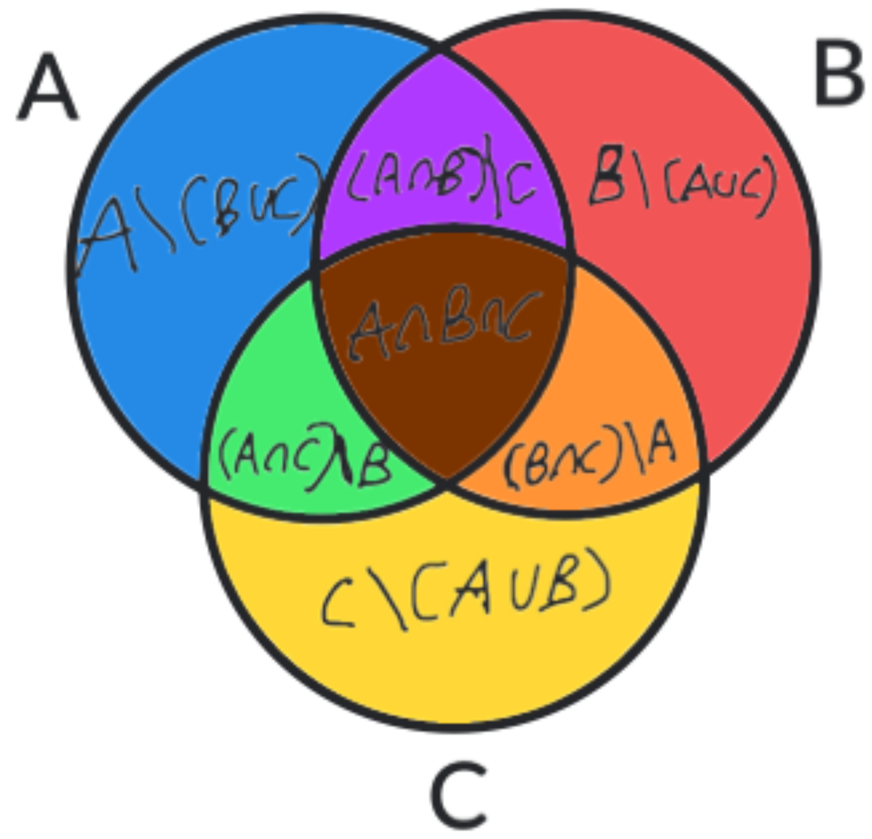
Similarly, $Pr(B) = Pr(B \setminus A) + Pr(A \cap B)$

$$Pr(A) + Pr(B) - Pr(A \cap B) = Pr(A \setminus B) + Pr(B \setminus A) + Pr(A \cap B) = Pr(A \cup B)$$

# Inclusion-Exclusion Principle for 3 Events

Inclusion-Exclusion Principle for 3 events:

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$$

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$$\Pr((A \cup B) \cup C) = \Pr(A \cup B) + \Pr(C) - \Pr((A \cup B) \cap C)$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Observe that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

$$-\Pr((A \cup B) \cap C) = -\Pr((A \cap C) \cup (B \cap C)) = -\Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$$

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$$

Challenge: What happens for $k$ events? Can you prove it?