Conditional Probability

Definition: Given events A and B, we define the conditional probability of A given B to be $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$.

Example: Let's say we flip a fair coin 3 times

A: The first coin is heads

B: At least 2 coins are heads

$|\Omega| = 2^3 = 8$

$B = \{HHH, HHT, HTH, THH\}$ so $Pr(B) = \frac{4}{8}$

$A \cap B = \{HHH, HHT, HTH\}$ so $Pr(A \cap B) = \frac{3}{8}$

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4} = \frac{|A \cap B|}{|B|}$$

Notes About Conditional Probability

Proposition: Given a probability space (Ω, Pr) and given an event B, $(B, Pr(\cdot|B))$ is also a probability space.

Proof: For each $w \in B$, $Pr(w|B) = \dfrac{Pr(w \cap B)}{Pr(B)} = \dfrac{Pr(w)}{Pr(B)}$

$$\sum_{w \in B} Pr(w|B) = \frac{\sum_{w \in B} Pr(w)}{Pr(B)} = \frac{Pr(B)}{Pr(B)} = 1$$

Proposition: If A and B are independent events then $Pr(A|B) = Pr(A)$

Proof: $Pr(A|B) = \dfrac{Pr(A \cap B)}{Pr(B)} = \dfrac{Pr(A)\cancel{Pr(B)}}{\cancel{Pr(B)}} = Pr(A).$

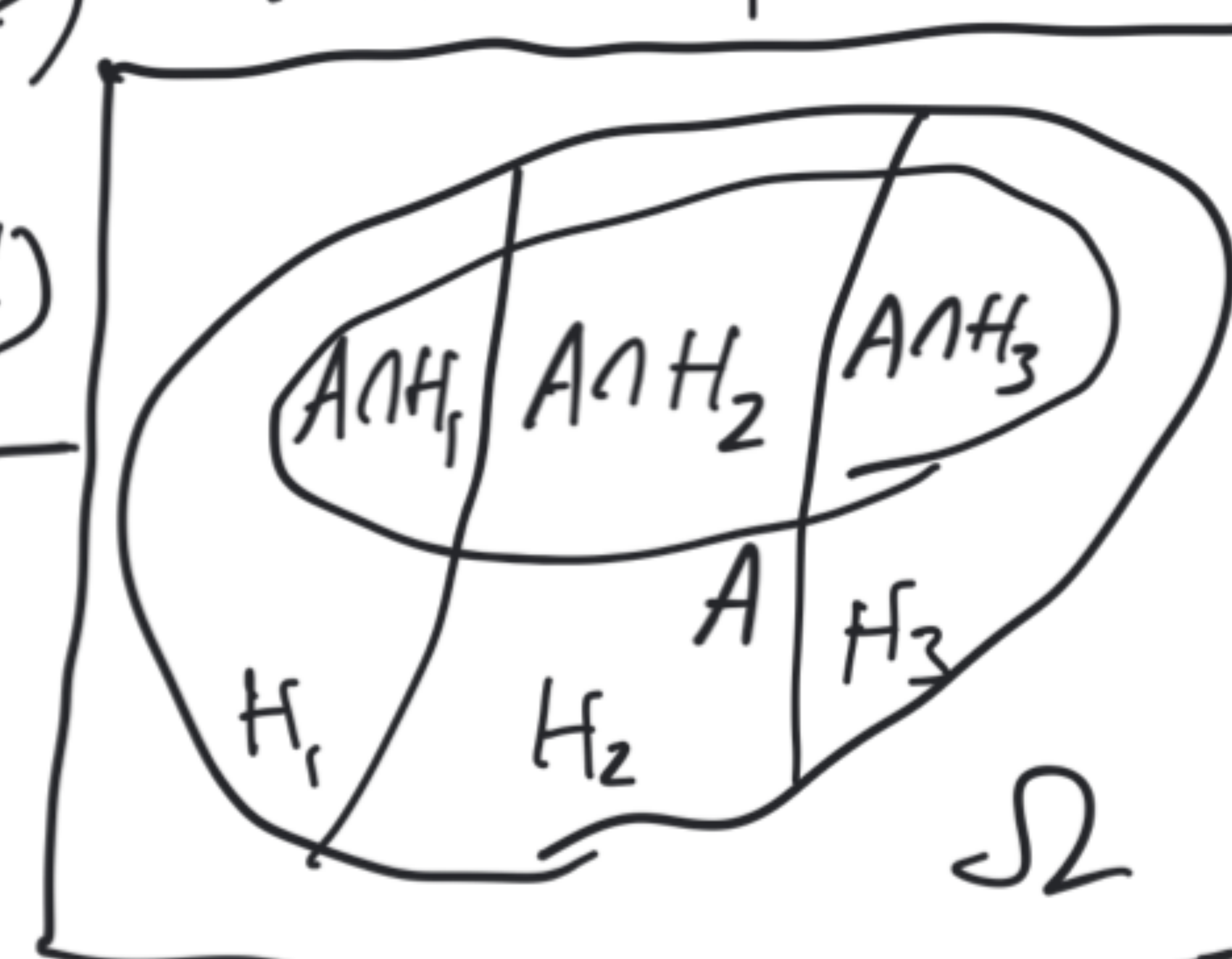
The Law of Total Probability

Law of Total Probability: Let $H_1, \dots, H_k$ be a partition of Ω. For any event A,

$$Pr(A) = \sum_{i=1}^{k} Pr(H_i) \cdot Pr(A \mid H_i)$$

Proof: $\forall i \in [k]$, $Pr(H_i) \cdot Pr(A \mid H_i) = \cancel{Pr(H_i)} \cdot \frac{Pr(A \cap H_i)}{\cancel{Pr(H_i)}} = Pr(A \cap H_i)$

$$\sum_{i=1}^{k} Pr(H_i) \cdot Pr(A \mid H_i) = \sum_{i=1}^{k} Pr(A \cap H_i) = Pr(A)$$



Example: If we roll 2 fair 6-sided dice, what is the probability that the numbers differ by at most 1.

H_i: The first die roll is i $\quad$ A: The numbers differ by at most 1

$\forall i \in [6]$, $Pr(H_i) = \frac{1}{6}$

$Pr(A \mid H_1) = \frac{2}{6}$, $Pr(A \mid H_2) = Pr(A \mid H_3) = Pr(A \mid H_4) = Pr(A \mid H_5) = \frac{3}{6}$, $Pr(A \mid H_6) = \frac{2}{6}$

$$Pr(A) = \frac{1}{6} \cdot \frac{2}{6} + 4 \cdot \frac{1}{6} \cdot \frac{3}{6} + \frac{1}{6} \cdot \frac{2}{6} = \frac{2+12+2}{36} = \frac{16}{36} = \frac{4}{9}$$