Probability Spaces

Definition: A probability space consists of a set Ω of atomic events and a map $Pr: \Omega \to [0,1]$ giving the probability of each atomic event $w \in \Omega$.
We require that $\sum_{w \in \Omega} Pr(w) = 1$.

Ω: Sample space
Pr: Probability distribution on Ω

Example: What will tomorrow's weather be?
$\Omega = \{$Sunny, Cloudy, Rainy, Snowy$\}$

$Pr(\text{Sunny}) = .4$, $Pr(\text{Cloudy}) = .3$, $Pr(\text{Rainy}) = .2$, $Pr(\text{Snowy}) = .1$

Uniform Distribution

Definition: Given a sample space Ω, the uniform distribution on Ω is the map $Pr: \Omega \to [0,1]$ such that $\forall \omega \in \Omega$, $Pr(\omega) = \frac{1}{|\Omega|}$.

In other words, the uniform distribution gives the same probability to each atomic event.

Example 1: If we roll a fair 6-sided die,

$\Omega = \{1,2,3,4,5,6\}$ $\quad Pr(1) = Pr(2) = Pr(3) = Pr(4) = Pr(5) = Pr(6) = \frac{1}{6}$

Example 2: If we flip a fair coin twice,

$\Omega = \{HH, HT, TH, TT\}$ $\quad Pr(HH) = Pr(HT) = Pr(TH) = Pr(TT) = \frac{1}{4}$

Probabilities Over the Uniform Distribution

Definition: Given a probability space (Ω, Pr), an event A is a subset $A \subseteq \Omega$.

$Pr(A) = \sum_{\omega \in A} Pr(\omega)$.

Example: If $\Omega = \{Sunny, Cloudy, Rainy, Snowy\}$ and $Pr(Sunny) = .4$ $Pr(Cloudy) = .3$, $Pr(Rainy) = .2$, $Pr(Snowy) = .1$

A: It will be rainy or snowy tomorrow.

$A = \{Rainy, Snowy\}$

$Pr(A) = Pr(Rainy) + Pr(Snowy) = .2 + .1 = .3$

Proposition: For the uniform distribution over a sample space Ω, for all events $A \subseteq \Omega$, $Pr(A) = \frac{|A|}{|\Omega|}$

Probabilities Over the Uniform Distribution

Example: If we roll a fair 6-sided die, what is the probability that we get 3 or higher?

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{3, 4, 5, 6\}$

$Pr(A) = \frac{|A|}{|\Omega|} = \frac{4}{6} = \frac{2}{3}$

Example: If we flip a fair coin 3 times, what is the probability that we get exactly 2 heads?

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$A = \{HHT, HTH, THH\}$

$Pr(A) = \frac{|A|}{|\Omega|} = \frac{3}{8}$