Disjoint Events

Definition: We say that two events A, B are disjoint if $A \cap B = \emptyset$.

Proposition: If A and B are disjoint then
$$Pr(A \cup B) = Pr(A) + Pr(B)$$

Example: Let's say we flip a fair coin 4 times.

A: We get exactly 1 heads

B: We get exactly 3 heads

$$Pr(A) = \frac{\binom{4}{1}}{2^4} = \frac{4}{16} = \frac{1}{4}$$

$$Pr(B) = \frac{\binom{4}{3}}{2^4} = \frac{4}{16} = \frac{1}{4}$$

$$Pr(A \cup B) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Independent Events

Definition: We say that two events A and B are independent if $Pr(A \cap B) = Pr(A)Pr(B)$.

Example: Let's say we flip a fair coin and roll a fair 6-sided die.

A: We get heads $\qquad Pr(A) = \frac{1}{2}$

B: The die roll is a 1 or a 2 $\qquad Pr(B) = \frac{2}{6} = \frac{1}{3}$

$Pr(A \cap B) = Pr(A)Pr(B) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

Note: It's possible for two events A and B to be related to each other but still be independent

Example: Let's say we roll a fair 6-sided die

A: We roll an odd number $\qquad$ B: We roll a 5 or 6

$Pr(A) = \frac{1}{2}$, $Pr(B) = \frac{1}{3}$ $\qquad Pr(A \cap B) = \frac{1}{6} = Pr(A)Pr(B)$

Disjointness and Independence for Several Events

Definition: We say that events $A_1, \ldots, A_k$ are disjoint if $\forall i \neq j \in [k]$, $A_i \cap A_j = \emptyset$.

If so, $Pr(A_1 \cup \ldots \cup A_k) = \sum_{i=1}^{k} Pr(A_i)$

Definition: We say that events $A_1, \ldots, A_k$ are independent if $\forall S \subseteq [k]$,

$$Pr\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} Pr(A_i)$$

Example: If $Pr(A) = \frac{1}{2}$, $Pr(B) = \frac{1}{3}$, and $Pr(C) = \frac{1}{4}$ then in order for A, B, C to be independent, we need that $Pr(A \cap B) = \frac{1}{6}$, $Pr(A \cap C) = \frac{1}{8}$, $Pr(B \cap C) = \frac{1}{12}$, $Pr(A \cap B \cap C) = \frac{1}{24}$.

Warning: It's possible for events A, B, C to be pairwise independent but not independent.