Functions

Definition: A function f from a set A to another set B is a map taking each element $a \in A$ to an element $f(a) \in B$.

$f : A \to B$ is shorthand for f is a function from A to B.

Definition: A function $f : A \to B$ is injective if $\forall a, a' \in A \, (a' \neq a \to f(a') \neq f(a))$

A function $f : A \to B$ is surjective if $\forall b \in B \, \exists a \in A \, (f(a) = b)$

Example: Let $A = \{-1, 0, 1\}$ and let $B = \{0, 1, 2\}$.

If we define $f : A \to B$ so that $f(x) = x + 1$ then f is both injective and surjective.

$$-1 \to 0$$
$$0 \to 1$$
$$1 \to 2$$
$$f$$

If we define $g : A \to B$ so that $g(x) = x^2$ then g is neither injective nor surjective.

$$-1 \to 1$$
$$0 \to 0$$
$$1 \to 1$$
$$g$$

Bijections

Definition: We say a function $f: A \to B$ is a bijection (or one-to-one) if it is both injective and surjective.

Example: If $A = \{-1, 0, 1\}$ and $B = \{0, 1, 2\}$ then $f(x) = x + 1$ is a bijection.

Example: If $A = \{0, 1, 2, 3\}$ and $B = \{0, 2, 4, 6\}$ then $g(x) = 2x$ is a bijection.

Proposition: If there is a bijection $f: A \to B$ then $|A| = |B|$.

To show $f: A \to B$ is a bijection, we generally give the inverse $f^{-1}: B \to A$ of f and confirm that

1) $\forall a \in A \, (f^{-1}(f(a)) = a)$

Note: This shows that f is injective because if $f(a') = f(a)$ then $a' = f^{-1}(f(a')) = f^{-1}(f(a)) = a$.

2) $\forall b \in B \, (f(f^{-1}(b)) = b)$

Note: This shows that f is surjective.

Examples: $f^{-1}(x) = x - 1$

$f^{-1}(f(x)) = (x+1) - 1 = x$

$f(f^{-1}(x)) = (x-1) + 1 = x$

$g^{-1}(x) = \frac{x}{2}$

$g^{-1}(g(x)) = \frac{2x}{2} = x$

$g(g^{-1}(x)) = 2 \cdot \frac{x}{2} = x$