Proof Strategies in Combinatorics

Q: How can we show that two expressions are the same?

Proof strategies:

1) Manipulate the expressions algebraically to show that they are equal.

2) Show that the two expressions are two ways of counting the same thing or that there is a bijection between the objects that the two expressions count.

Example: $\binom{n}{n-k} = \binom{n}{k}$

Proof 1: $\binom{n}{n-k} = \dfrac{n!}{(n-k)!\,(n-(n-k))!} = \dfrac{n!}{(n-k)!\,k!} = \binom{n}{k}$

Proof 2:

$\binom{n}{n-k} = |\{S : S \subseteq [n], |S| = n-k\}|$

$\binom{n}{k} = |\{S : S \subseteq [n], |S| = k\}|$

$\{S : S \subseteq [n], |S| = n-k\}$

$\updownarrow$ $f(S) = [n] \setminus S$, $f^{-1} = f$

$\{S : S \subseteq [n], |S| = k\}$

Combinatorics Proof Examples

Example: For all non-negative integers n, k_1, k_2 such that $k_1 + k_2 \le n$, $\binom{n}{k_1}\binom{n-k_1}{k_2} = \binom{n}{k_2}\binom{n-k_2}{k_1}$

Proof one:

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!\cancel{(n-k_1)!}} \cdot \frac{\cancel{(n-k_1)!}}{k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!(n-k_1-k_2)!}$$

$$\binom{n}{k_2}\binom{n-k_2}{k_1} = \frac{n!}{k_2!\cancel{(n-k_2)!}} \cdot \frac{\cancel{(n-k_2)!}}{k_1!(n-k_2-k_1)!} = \frac{n!}{k_1!k_2!(n-k_1-k_2)!}$$

Proof two: $\binom{n}{k_1}\binom{n-k_1}{k_2}$ and $\binom{n}{k_2}\binom{n-k_2}{k_1}$ both count the # of ways to partition $[n]$ into 3 sets S_1, S_2, and S_3 with cardinalities k_1, k_2, and $n-k_1-k_2$, respectively.

$\binom{n}{k_1}\binom{n-k_1}{k_2}$: First choosing S_1 and then choosing S_2 from the remaining $n-k_1$ elements (taking $S_3 = [n] \setminus (S_1 \cup S_2)$).

$\binom{n}{k_2}\binom{n-k_2}{k_1}$: First choosing S_2 and then choosing S_1 from the remaining $n-k_2$ elements (taking $S_3 = [n] \setminus (S_1 \cup S_2)$).