Catalan Numbers

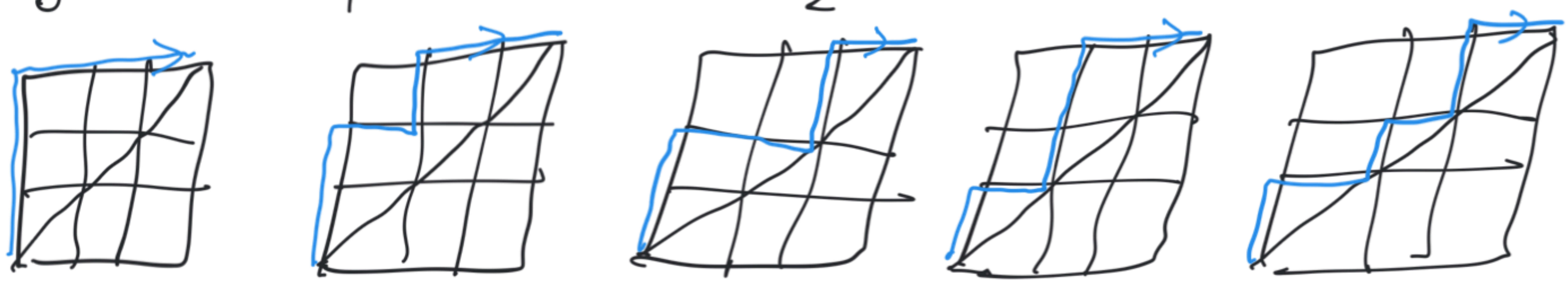
Definition: The Catalan number C_n is the number of ways to go from the lower left of an $n \times n$ grid to the upper right without going below the main diagonal.

Examples:

$C_0 = 1$

$C_1 = 1$

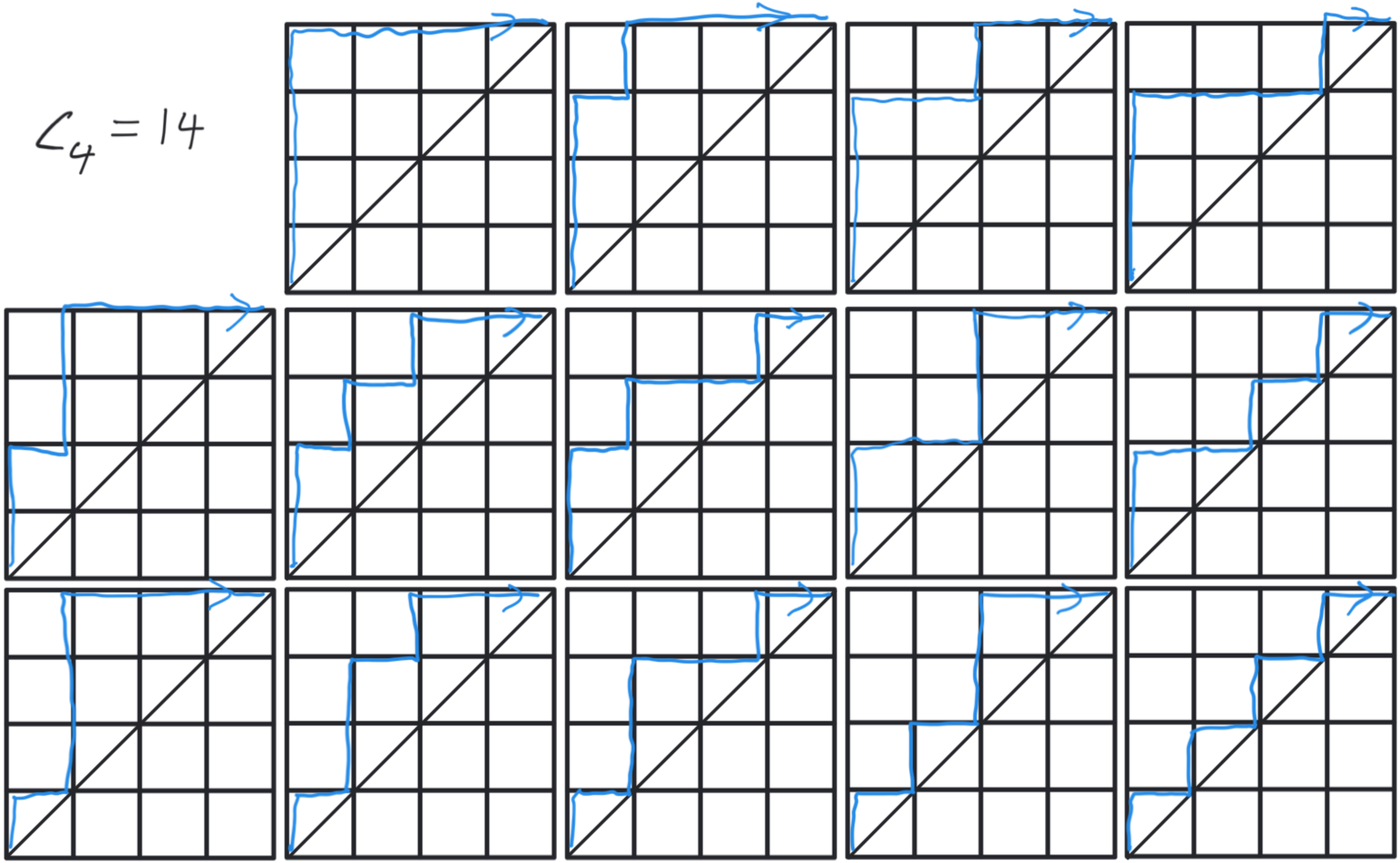
$C_2 = 2$



$C_3 = 5$

Catalan Numbers

$C_4 = 14$



Explicit Formula for the Catalan Numbers

Theorem: $C_n = \frac{1}{n+1}\binom{2n}{n}$

Examples:

$C_0 = \frac{1}{1}\cdot\binom{0}{0} = \frac{1}{1} = 1$ $\quad C_1 = \frac{1}{2}\binom{2}{1} = \frac{2}{2} = 1$ $\quad C_2 = \frac{1}{3}\binom{4}{2} = \frac{6}{3} = 2$

$C_3 = \frac{1}{4}\binom{6}{3} = \frac{20}{4} = 5$ $\quad C_4 = \frac{1}{5}\binom{8}{4} = \frac{\cancel{8}\cdot 7\cdot \overset{2}{\cancel{6}}\cdot\cancel{5}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 14$ $\quad C_5 = \frac{1}{6}\binom{10}{5} = \frac{\overset{2}{\cancel{10}}\cdot\overset{3}{\cancel{9}}\cdot\cancel{8}\cdot 7\cdot\cancel{6}}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = 42$

Proof:

Lemma: The number of walks on an $n \times n$ grid from the lower left to the upper right which go below the main diagonal is $\binom{2n}{n-1}$.

Total # of walks from the lower left to the upper right: $\binom{2n}{n}$

$(n-1)! = \frac{n!}{n}$ $\quad (n+1)! = (n+1)(n!)$

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{n!n!} - \frac{n}{n+1}\cdot\frac{(2n)!}{n!n!} = \left(1 - \frac{n}{n+1}\right)\frac{(2n)!}{n!n!}$$

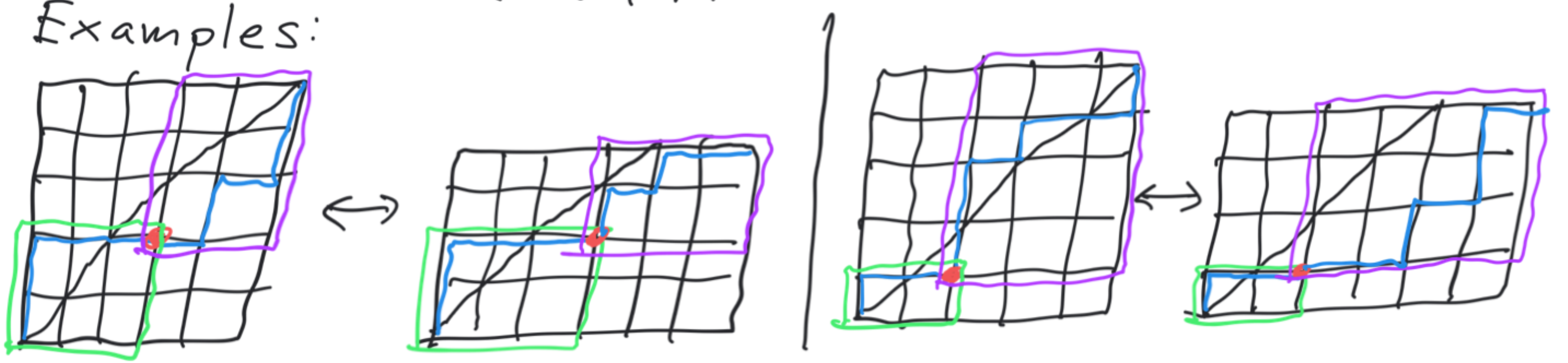
$$= \frac{1}{n+1}\binom{2n}{n}$$

Explicit Formula for the Catalan Numbers

Lemma: The number of walks on an $n \times n$ grid from the lower left to the upper right which go below the main diagonal is $\binom{2n}{n-1}$.

Proof: We have the following bijection between walks which go below the main diagonal and walks on an $(n-1)\times(n+1)$ grid:
Take the first point where the walk goes below the main diagonal and flip the remainder of the walk.

Examples:



Recurrence Relation for the Catalan Numbers

Theorem: $\forall n \in \mathbb{N} \left(C_n = \sum_{j=1}^{n} C_{j-1} C_{n-j} \right)$

Examples:

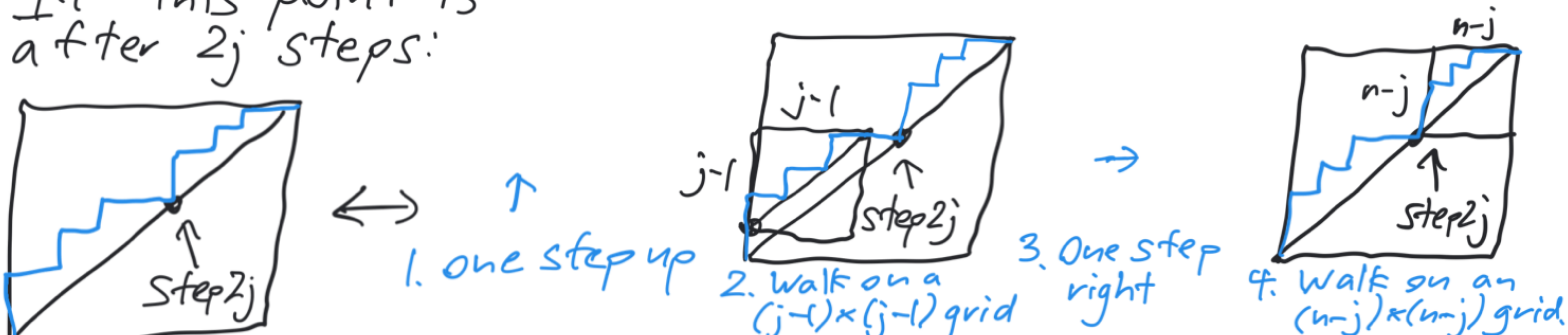
Recall $C_0 = 1$

$C_1 = C_0 \cdot C_0 = 1$, $C_2 = \overset{1\cdot 1 + 1\cdot 1}{C_0 \cdot C_1 + C_1 \cdot C_0} = 2$, $C_3 = \overset{1\cdot 2 + 1\cdot 1 + 2\cdot 1}{C_0 \cdot C_2 + C_1 \cdot C_1 + C_2 \cdot C_0} = 5$

$C_4 = \overset{1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1}{C_0 \cdot C_3 + C_1 \cdot C_2 + C_2 \cdot C_1 + C_3 \cdot C_0} = 14$, $C_5 = \overset{1\cdot 14 + 1\cdot 5 + 2\cdot 2 + 5\cdot 1 + 14\cdot 1}{C_0 \cdot C_4 + C_1 \cdot C_3 + C_2 \cdot C_2 + C_3 \cdot C_1 + C_4 \cdot C_0} = 42$

Proof idea: Partition the walks based on the first point where they return to the main diagonal.

If this point is after $2j$ steps:



1. one step up
2. Walk on a $(j-1)\times(j-1)$ grid
3. One step right
4. Walk on an $(n-j)\times(n-j)$ grid

# of choices for walks which first return to the main diagonal after $2j$ steps: $1 \times C_{j-1} \times 1 \times C_{n-j}$