Unlabeled Balls in Labeled Bins Examples

Q: How many ways are there to put n unlabeled balls into k labeled bins?

Trick: This is equivalent to placing k-1 dividing lines among the n balls.

There are $\binom{n+k-1}{k-1}$ ways to do this.

Several kinds of questions are equivalent to putting unlabeled balls in labeled bins.

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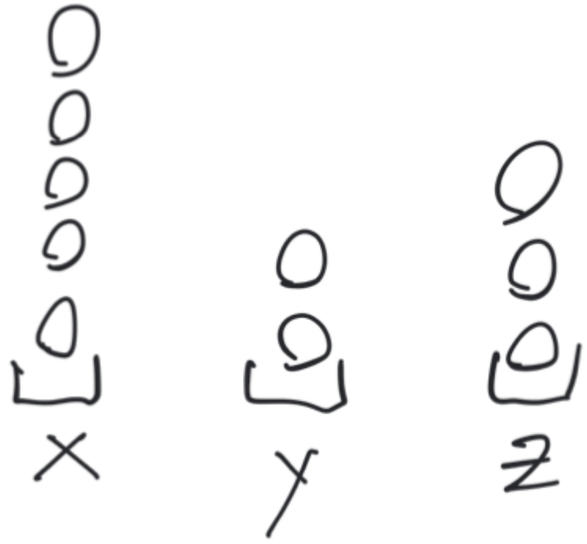
Q: How many triples of non-negative integers x,y,z are there such that x+y+z=10?

This is equivalent to putting 10 unlabeled balls into 3 bins, x, y, and z.

Example: $x=5, y=2, z=3 \longleftrightarrow$



$n = 10$ balls

$k = 3$ bins

Answer: $\binom{n+k-1}{k-1} = \binom{12}{2} = \frac{12 \cdot 11}{2 \cdot 1} = 66$

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Q: If a donut store sells 6 different kinds of donuts, how many different ways are there to buy 4 donuts from the store?

Ball: Order for 1 donut

Bin: Kind of donut

$n = 4$ balls

$k = 6$ bins

Answer: $\binom{n+k-1}{k-1} = \binom{9}{5} = \binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 63 \cdot 2 = 126$

Unlabeled Balls in Labeled Bins Examples

Q: If we have 3 distinguishable dice, how many different ways are there to roll the dice and get a sum of 8? How about 9?

This is equivalent to putting 8 unlabeled balls in 3 labeled bins except that each bin must have at least one ball.

After placing one ball in each bin, we choose where the remaining 5 balls go.

$n = 5$ balls
$k = 3$ bins

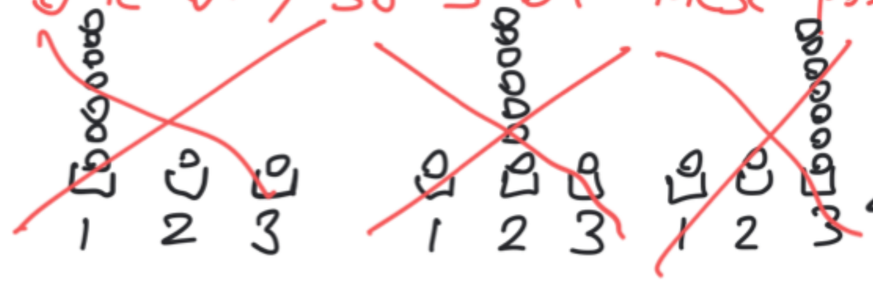
Answer: $\binom{n+k-1}{k-1} = \binom{7}{2} = \frac{7 \cdot \cancel{6}^{3}}{\cancel{2} \cdot 1} = 21$

If the sum is 9,

$n = 6$ balls
$k = 3$ bins

$\binom{n+k-1}{k-1} = \binom{8}{2} = \frac{\cancel{8}^{4} \cdot 7}{\cancel{2} \cdot 1} = 28$

We can't have more than 6 balls in one bin, so 3 of these possibilities are invalid.



Answer:

$28 - 3 = 25$