Combinatorics Examples

General techniques:

1. Consider the problem step by step and determine the number of choices for each step.

2. Sometimes there will be more than one way to get the same answer, in which case we are double-counting. To correct for this, we divide by the number of ways to get the same answer.

$$\#\,of\ answers = \frac{total\ \#\ of\ choices}{\#\ of\ ways\ to\ get\ the\ same\ answer.}$$

Combinatorics Examples

Q: How many 4-digit numbers are there whose digits are all distinct and nonzero?

Answer:

$$9 \cdot 8 \cdot 7 \cdot 6 = \boxed{3024}$$

1st digit · 2nd digit · 3rd digit · 4th digit

Combinatorics Examples

Q: How many 5-card poker hands are there which have two pairs and one other card?

Answer:

$$13 \cdot \binom{12}{2} \cdot 4 \cdot \binom{4}{2} \cdot \binom{4}{2}$$

singleton value — values of the two pairs — suit of the singleton — suits of the higher pair — suits of the lower pair

$$\binom{4}{2} = 6$$

Answer: $13 \cdot \dfrac{12 \cdot 11}{2} \cdot 4 \cdot 6 \cdot 6 = \boxed{123552}$

Combinatorics Examples

Q: How many different ways are there to rearrange the letters of the word banana?

Answer:

$$\binom{6}{3} \qquad \cdot \qquad \binom{3}{2} \qquad \cdot \quad 1 \qquad = 20 \cdot 3 = \boxed{60}$$

$\begin{bmatrix}$ positions of the three a

positions of the two n

position of b

total letters

Observation: $\binom{6}{3} \cdot \binom{3}{2} = \dfrac{6!}{3! \, 3!} \cdot \dfrac{3!}{2! \, 1!} = \dfrac{6!}{3! \, 2! \, 1!}$

$a \; n \; b$

Alternative reasoning: Consider permutations of the 6 positions.

$$\dfrac{6!}{3! \cdot 2!} \quad \begin{array}{l} \leftarrow \text{total \# of permutations} \\ \leftarrow \text{\# of permutations giving the same answer} \end{array}$$

Combinatorics Examples

Q: How many different ways are there to divide 6 people into two teams of 3?

First attempt: Choose one team of 3. This will determine the other team.

There are $\binom{6}{3} = 20$ different ways to choose a team of 3.

Issue: We are double-counting!

If the people are A, B, C, D, E, and F, choosing the team ABC and choosing the team DEF both give the split ABC/DEF.

Answer: $\dfrac{\binom{6}{3}}{2}$ ← # of ways to pick a team of 3

← # of ways to get the same split

$= \dfrac{20}{2} = \boxed{10}$