Pascal's Triangle, Binomial Coefficients, and the Binomial Theorem

Discrete Mathematics 27100 Winter 2022

1 Pascal's Triangle

Definition 1.1. *For all non-negative integers n and all integers k such that $0 \leq k \leq n$, we define the entry $P(n,k)$ in row n and column k of Pascal's triangle so that*

1. *For all non-negative integers n, $P(n,0) = P(n,n) = 1$*
2. *Whenever $0 < k < n$, $P(n,k) = P(n-1,k-1) + P(n-1,k)$*

Visually, Pascal's triangle can be depicted as follows (yes, it's prettier as an equilateral triangle rather than a right triangle)

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

There are many patterns in Pascal's triangle. A few of them are as follows:

1. For all non-negative integers n and all integers k such that $0 \leq k \leq n$, $P(n,k) = P(n, n-k)$.
2. For all $n \geq 0$, $P(n,0) = 1$ (the zeroth column is all ones)
3. For all $n \geq 1$, $P(n,1) = n$ (the first column is the row number)
4. For all $n \geq 2$, $P(n,2) = \frac{n(n-1)}{2}$ (the second column is the triangular numbers)
5. The sum of the numbers in row n is 2^n.
6. The entry in row n and column k is the number of ways to reach row n and column k by starting at row 0 and column 0 and going either one row down or one row down and one column to the right at each step.

2 Binomial Coefficients

The following fundamental question is used throughout combinatorics:
Q: How many ways are there to choose a set of k objects out of a set of n objects?

Answer: If we choose the k objects one at a time then there are n choices for the first object, $n-1$ choices for the second object, ..., and $n-k+1$ choices for the kth object. Thus, the total number of choices is $\prod_{j=0}^{k-1}(n-j) = \frac{n!}{(n-k)!}$

However, this counts each set of k objects multiple times as there are multiple choices which result in the same set of k objects. For example, if we are choosing 3 numbers out of $\{1,2,3,4,5\}$ then choosing 2, 4, and 5 and choosing 5, 4, and 2 both result in the set $\{2,4,5\}$. To handle this, we use the following general combinatorial principle:

Principle 2.1.

$$\textit{\# of answers} = \frac{\textit{total \# of choices}}{\textit{\# of ways to choose the same answer}}$$

Here there are $k!$ different ways to choose the same set of k objects so the number of ways to choose a set of k objects out of a set of n objects is $\frac{n!}{k!(n-k)!}$.

This question appears so often as a subproblem of other problems that we have specific notation for $\frac{n!}{k!(n-k)!}$.

Definition 2.2. *Given integers n and k such that $n \geq 0$ and $0 \leq k \leq n$, we define $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to be the number of ways to choose k objects out of n objects.*

Example 2.3.

1. $\binom{5}{2} = \frac{5!}{2!3!} = \frac{5*4}{2*1} = 10$
2. $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6*5*4}{3*2*1} = 20$

2.1 Using Binomial Coefficients

Binomial coefficients are ubiquitous in combinatorics. Two simple examples are as follows

Example 2.4. *Q: How many binary strings (strings of zeros and ones) are there of length 8 which have exactly 3 ones?*

Answer: This is just $\binom{8}{3} = \frac{8*7*6}{3*2*1} = 56$

Example 2.5. *Q: How many ternary strngs (strings of zeros, ones, and twos) are there of length 6 which have exactly 2 twos?*

Answer: To choose a ternary string of length 6 with exactly 2 twos, we must make the following choices:

1. *Which 2 characters are twos?*
2. *Are the remaining 4 characters zeros or ones?*

There are $\binom{6}{2} = \frac{6*5}{2*1} = 15$ choices for the locations of the 2 twos and there are $2^4 = 16$ choices for whether the remaining four characters are zero or one so the total number of choices is $15*16 = 240$

2.2 Binomial Coefficients and Pascal's Triangle

The entries of Pascal's triangle are in fact the binomial coefficients and we now prove this.

Lemma 2.6. *For all integers n and k such that $n \geq 1$ and $0 < k < n$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$*

Proof. We can see this from the following combinatorial argument. In order to choose k out of n objects, we can either first choose $k-1$ out of the first $n-1$ objects and then take the last object or we can choose k out of the first $n-1$ objects and not take the last object.

We can also see this by explcitly computing $\binom{n-1}{k-1} + \binom{n-1}{k}$. Observe that

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{k*(n-1)!}{k!(n-k)!} + \frac{(n-k)*(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}\end{aligned}$$

□

Theorem 2.7. *For all integers n and k such that $n \geq 0$ and $0 \leq k \leq n$, the entry $P(n,k)$ in row n and column k of Pascal's triangle is $P(n,k) = \binom{n}{k}$*

Proof. We can prove this by induction on n. For the base case $n = 0$, by definition, $P(0,0) = 1 = \binom{0}{0}$. For the inductive step, assume that the result is true for $n = j-1$ and consider $n = j$. By definition, $P(n,0) = 1 = \binom{n}{0}$ and $P(n,n) = 1 = \binom{n}{n}$. For all $k \in [n-1]$, $P(n,k) = P(n-1,k-1) + P(n-1,k)$. By the inductive hypothesis, $P(n-1,k-1) = \binom{n-1}{k-1}$ and $P(n-1,k) = \binom{n-1}{k}$. By Lemma 2.6, $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. Putting everything together,

$$P(n,k) = P(n-1,k-1) + P(n-1,k) = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

as needed. □

3 The Binomial Theorem

Theorem 3.1 (The Binomial Theorem)**.** *For all $n \in \mathbb{N} \cup \{0\}$,*

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Proof. To see why the binomial theorem is true, write $(x+y)^n = (x+y)*(x+y)*\ldots*(x+y)$ and observe that to get an $x^{n-j}y^j$ term, we have to pick j out of n y. There are $\binom{n}{j}$ ways to do this so the coefficient of $x^{n-j}y^j$ in $(x+y)^n$ is $\binom{n}{j}$.

We can also prove the binomial theorem by induction. For the base case $n = 0$, $(x+y)^0 = 1 = x^0y^0$. For the inductive step, assume that the result is true for $n = k-1$ and consider $n = k$. By the inductive hypothesis, $(x+y)^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-1-j} y^j$ so

$$\begin{aligned}
(x+y)^n = (x+y) * (x+y)^{n-1} &= (x+y)\left(\sum_{j=0}^{n-1}\binom{n-1}{j} x^{n-1-j}y^j\right) \\
&= x\left(\sum_{j=0}^{n-1}\binom{n-1}{j} x^{n-1-j}y^j\right) + y\left(\sum_{j=0}^{n-1}\binom{n-1}{j} x^{n-1-j}y^j\right) \\
&= \sum_{j=0}^{n-1}\binom{n-1}{j} x^{n-j}y^j + \sum_{j=0}^{n-1}\binom{n-1}{j} x^{n-1-j}y^{j+1} \\
&= \sum_{j=0}^{n-1}\binom{n-1}{j} x^{n-j}y^j + \sum_{j=1}^{n}\binom{n-1}{j-1} x^{n-j}y^j \\
&= x^n + \sum_{j=1}^{n-1}\left(\binom{n-1}{j} + \binom{n-1}{j-1}\right) x^{n-j}y^j + y^n \\
&= \sum_{j=0}^{n}\binom{n}{j} x^{n-j}y^j
\end{aligned}$$

$\square$