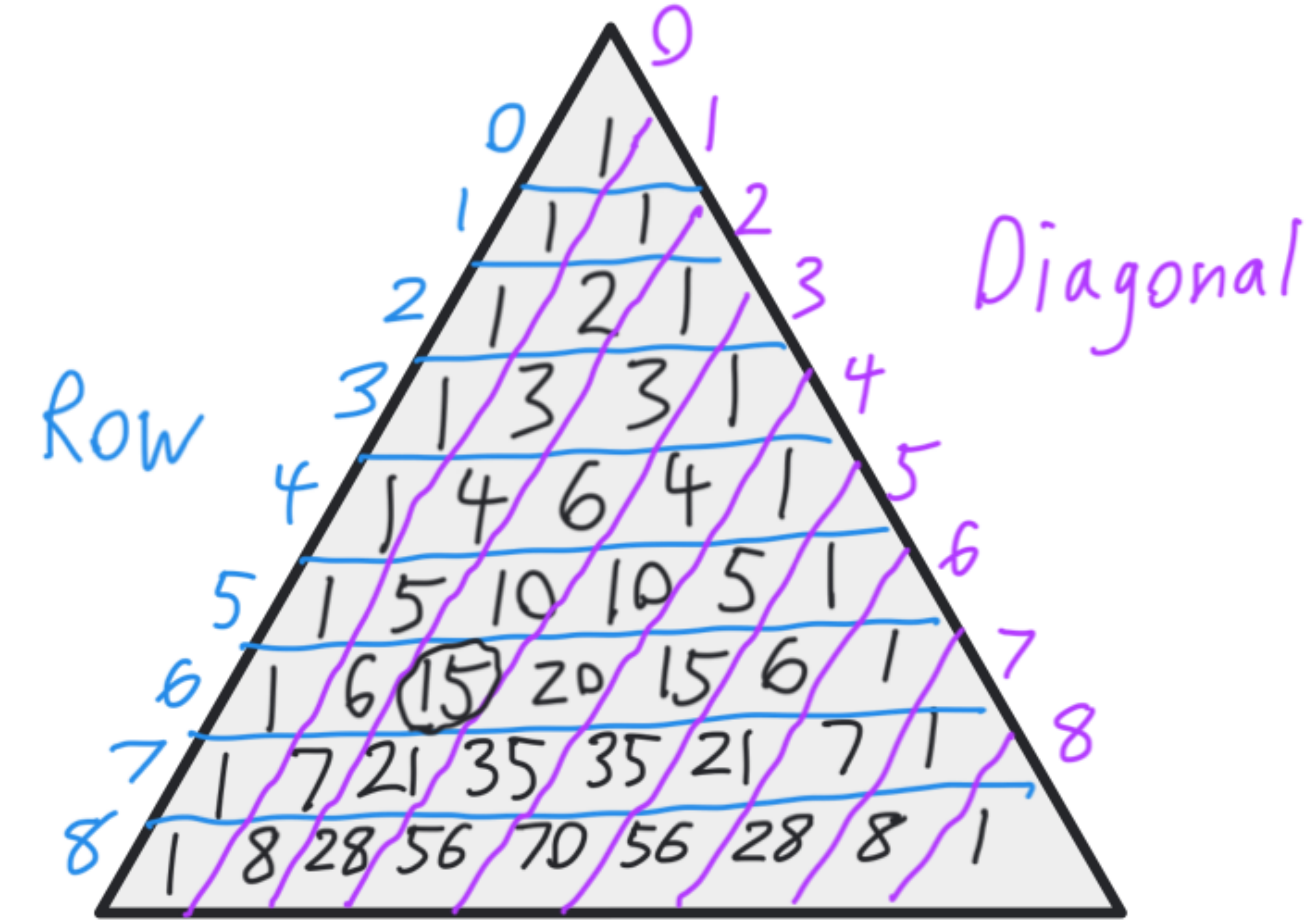
Binomial Coefficients in Pascal's Triangle



Let $P(n,j)$ be the entry in row n and diagonal j.

Claim: $P(n,j) = \binom{n}{j}$.

Example: $\binom{6}{2} = \frac{6\cdot 5}{2\cdot 1} = 3\cdot 5 = 15 = P(6,2)$

One explanation: $P(n,j)$ is the # of paths from the top down to the entry in row n and diagonal j.
Choosing such a path is equivalent to choosing j out of n steps to go right.

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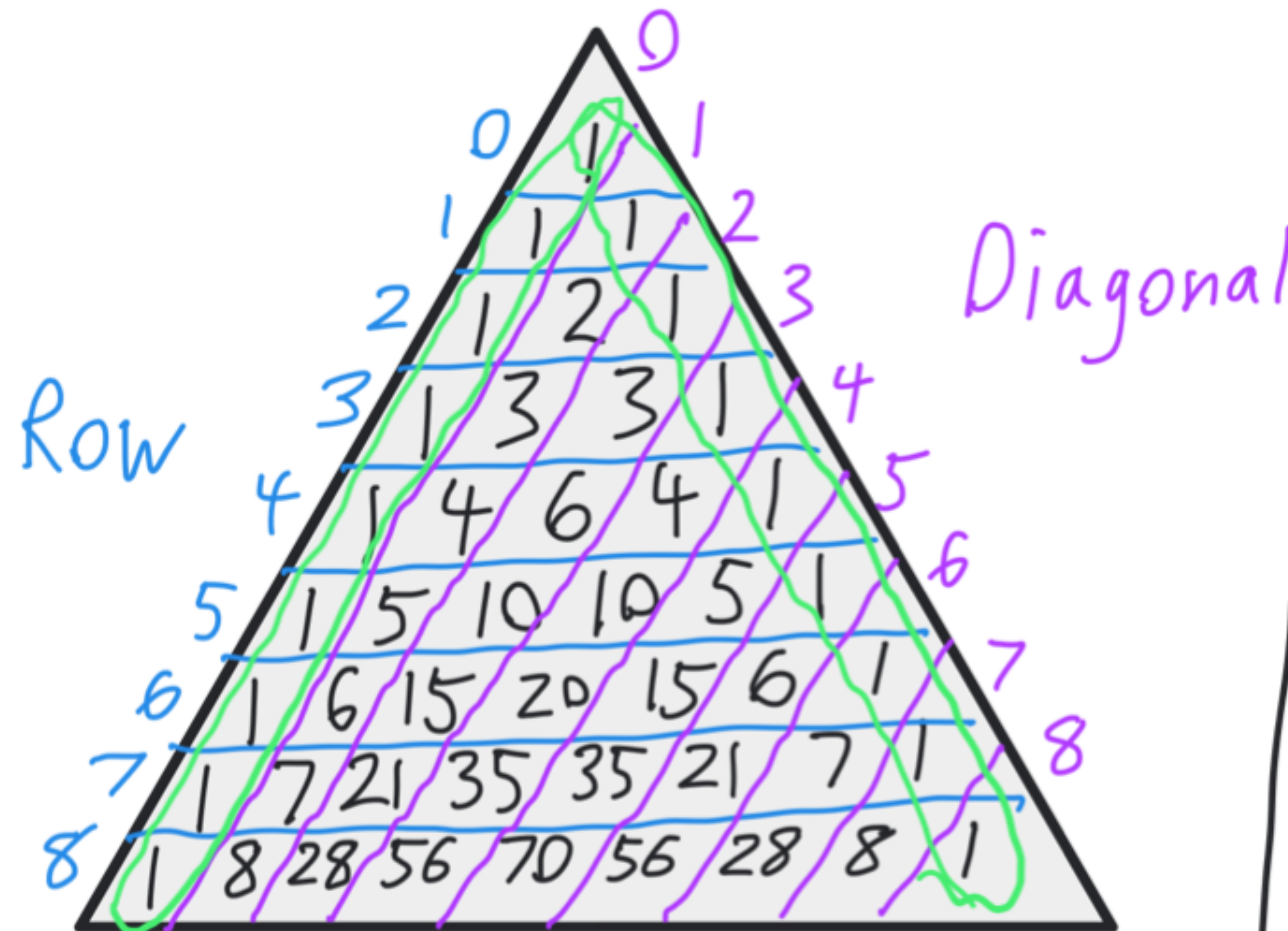

Proof of claim:

Base cases:

$P(0,0) = \binom{0}{0} = 1$

$\forall n \in \mathbb{N} \left(P(n,0) = \binom{n}{0} = 1 \right)$

$\forall n \in \mathbb{N} \left(P(n,n) = \binom{n}{n} = 1 \right)$

Inductive step:

Assume $\forall j \in \{0,1,\dots,k\} \left(P(k,j) = \binom{k}{j} \right)$ and consider $n = k+1$ and $j \in [k]$.

$$P(k+1,j) = P(k,j-1) + P(k,j) = \binom{k}{j-1} + \binom{k}{j} = \binom{k+1}{j} \checkmark$$

Claim: $\forall n \in \mathbb{N} \cup \{0\}\ \forall j \in \{0,1,\dots,n\} \left(P(n,j) = \binom{n}{j} \right)$

Key Lemma: $\forall n \in \mathbb{N}\ \forall j \in [n] \left(\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1} \right)$

Proof 1: To choose j out of $n+1$ objects, we can either take the last object and choose $j-1$ out of the first n objects or not take the last object and choose j out of the first n objects.

Proof 2: $$\binom{n}{j} + \binom{n}{j-1} = \frac{n!}{j!(n-j)!} + \frac{n!}{(j-1)!(n-j+1)!} = \left(1 + \frac{j}{n-j+1}\right) \cdot \frac{n!}{j!(n-j)!} = \frac{n+1}{n-j+1} \cdot \frac{n!}{j!(n-j)!} = \frac{(n+1)!}{j!(n-j+1)!} = \binom{n+1}{j}$$