Binomial Theorem

Binomial Theorem: $\forall n \in \mathbb{N} \cup \{0\} \left((x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \right)$

Examples:

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

etc.

Binomial Theorem Proof

Binomial Theorem: $\forall n \in \mathbb{N} \cup \{0\} \left((x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \right)$

Proof: By induction.

Base cases: If $n=0$, $(x+y)^0 = 1 = \binom{0}{0} \cdot x^0 \cdot y^0$

If $n=1$, $(x+y)^1 = x+y = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1$

Inductive step: Assume $(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j$ and consider $n=k+1$.

$$(x+y)^{k+1} = (x+y)(x+y)^k = (x+y)\left(\sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j\right)$$

$$= x\left(\sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j\right) + y\left(\sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j\right)$$

$$= \left(\sum_{j=0}^{k} \binom{k}{j} x^{(k+1)-j} y^j\right) + \left(\sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^{j+1}\right)$$

Trick: Let $j = j'-1$

$$= \left(\sum_{j=0}^{k} \binom{k}{j} x^{(k+1)-j} y^j\right) + \left(\sum_{j'=1}^{k+1} \binom{k}{j'-1} x^{(k+1)-j'} y^{j'}\right)$$

$$= \binom{k}{0} x^{k+1} + \left(\sum_{j=1}^{k} \left(\binom{k}{j} + \binom{k}{j-1}\right) x^{(k+1)-j} y^j\right) + \binom{k}{k} y^{k+1}$$

$$= \binom{k+1}{0} x^{k+1} + \left(\sum_{j=1}^{k} \binom{k+1}{j} x^{(k+1)-j} \cdot y^j\right) + \binom{k+1}{k+1} \cdot y^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} x^{(k+1)-j} y^j$$

Recall: $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$