Binomial Coefficients

Q: How many ways are there to choose a sequence of k distinct numbers from $[n]$?

Example: If $n=4$ and $k=2$,

1,2	2,1	3,1	4,1
1,3	2,3	3,2	4,2
1,4	2,4	3,4	4,3

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General n and k:

$$\underbrace{n \text{ choices}}_{\text{1st number}} \times \underbrace{(n-1) \text{ choices}}_{\text{2nd number}} \times \cdots \times \underbrace{(n-k+1) \text{ choices}}_{k\text{th number}}$$

Answer: $n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \prod_{j=n-k+1}^{n} j = \frac{n!}{(n-k)!}$

Recall that $n! = \prod_{j=1}^{n} j = n \cdot (n-1) \cdot \ldots \cdot 1$

$$\cdot \ \frac{n!}{(n-k)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \cdot \cancel{(n-k) \cdot \ldots \cdot 1}}{\cancel{(n-k) \cdot \ldots \cdot 1}}$$

$$= n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$$

Special case: There are $n!$ orderings of $[n]$.

6 orderings of $[3]$:

1,2,3	2,1,3	3,1,2
1,3,2	2,3,1	3,2,1

24 orderings of $[4]$:

1,2,3,4	2,1,3,4	3,1,2,4	4,1,2,3
1,2,4,3	2,1,4,3	3,1,4,2	4,1,3,2
1,3,2,4	2,3,1,4	3,2,1,4	4,2,1,3
1,3,4,2	2,3,4,1	3,2,4,1	4,2,3,1
1,4,2,3	2,4,1,3	3,4,1,2	4,3,1,2
1,4,3,2	2,4,3,1	3,4,2,1	4,3,2,1

Binomial Coefficients

Definition: The binomial coefficient $\binom{n}{k}$ (which we can call n choose k) is the number of ways to choose a set of k out of n objects.

Example: $\binom{4}{2} = 6$ Ways to choose 2 out of 4 objects:
$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$

Theorem: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k\cdot(k-1)\cdot\ldots\cdot 1}$

Proof: Let's say we choose a sequence of k distinct objects. This gives us a set of k objects, but there may be multiple ways to obtain the same set. Example: 2,4 and 4,2 both give $\{2,4\}$.

Combinatorial rule: $\#$ of answers $= \dfrac{\text{total \# of choices}}{\text{\# of ways to obtain a given answer}}$ assuming this is the same for all answers.

$\#$ of ways to choose a sequence of k distinct objects: $\dfrac{n!}{(n-k)!} = n\cdot(n-1)\cdot\ldots\cdot(n-k+1)$

$\#$ of ways to choose a given set $=$ $\#$ of orderings of k objects $= k! = k\cdot(k-1)\cdot\ldots\cdot 1$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k\cdot(k-1)\cdot\ldots\cdot 1}$$

Binomial Coefficients Examples

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k\cdot(k-1)\cdot\ldots\cdot 1}$$

Examples:

$$\binom{5}{2} = \frac{5!}{2!3!} = \frac{120}{2\cdot 6} = 10 \qquad \binom{5}{2} = \frac{5\cdot 4}{2\cdot 1} = \frac{20}{2} = 10$$

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6\cdot 6} = 20 \qquad \binom{6}{3} = \frac{\cancel{6}\cdot 5\cdot 4}{\cancel{3}\cdot\cancel{2}\cdot 1} = 20$$

$$\binom{9}{4} = \frac{9\cdot 8\cdot 7\cdot\cancel{6}}{4\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = \frac{8}{4}\cdot 9\cdot 7 = 2\cdot 9\cdot 7 = 126$$

Proposition: $\binom{n}{n-k} = \binom{n}{k}$

Proof: $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$

Edge cases: $0! = 1$

$$\binom{n}{0} = \frac{n!}{0!n!} = \frac{n!}{n!} = 1 \qquad \binom{n}{n} = \frac{n!}{n!0!} = \frac{n!}{n!} = 1$$

If $k<0$ or $k>n$, we define $\binom{n}{k} = 0$ as there are zero ways to choose k out of n objects.