Introduction to Combinatorics and the Inclusion Exclusion Principle

Discrete Mathematics 27100 Winter 2022

1 Basic Combinatorial Principles

We start with some basic combinatorial principles.

A first basic but very useful observation is that if two (or more) sets are disjoint from each other then the number of elements in their union is equal to the sum of the number of elements in each set.

Proposition 1.1. *If A and B are disjoint sets then $|A \cup B| = |A| + |B|$. More generally, given a natural number $k \in \mathbb{N}$ and finite sets $S_1, \ldots, S_k$ such that $\forall i, j \in [k] : i \neq j, S_i \cap S_j = \emptyset$,*

$$\left|\cup_{i=1}^{k} S_i\right| = \sum_{i=1}^{k} |S_i|$$

Example 1.2. *If $A = \{0, 2\}$ and $B = \{3, 5, 7\}$ then $A \cup B = \{0, 2, 3, 5, 7\}$. Here $|A| = 2$, $|B| = 3$, and $|A \cup B| = 5$.*

Based on this, we have the following combinatorial principle:

Principle 1.3. *If there are disjoint cases then the total number of possibilities is equal to the sum of the number of possibilities for each case.*

A second basic but very useful observation is that the number of elements in the cartesian product of two (or more) sets is equal to the product of the number of elements in each set.

Proposition 1.4. *Given finite sets A and B, $|A \times B| = |A| * |B|$. More generally, given a natural number $k \in \mathbb{N}$ and finite sets $S_1, \ldots, S_k$,*

$$|S_1 \times S_2 \times \ldots \times S_k| = \prod_{i=1}^{k} |S_i|$$

Example 1.5. *If $A = \{0, 2\}$ and $B = \{3, 5, 7\}$ then $A \times B = \{(0, 3), (0, 5), (0, 7), (2, 3), (2, 5), (2, 7)\}$. Here $|A| = 2$, $|B| = 3$, and $|A \times B| = 6$.*

Based on this, we have the following combinatorial principle:

Principle 1.6. *If we have independent choices, the total number of possibilities is equal to the product of the number of possibilities for each choice.*

Definition 1.7. *Given a set S, we define the power set $P(S)$ of S to be $P(S) = \{S' : S' \subseteq S\}$. In other words, $P(S)$ is the set of all subsets of S.*

Example 1.8. *If $S = \{1, 2, 3\}$ then $P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$*

Lemma 1.9. *For all finite sets S, $|P(S)| = 2^{|S|}$*

Proof. To choose a subset S' of S, we need to choose whether each element $s \in S$ is in S'. For each of these choices, there are 2 possibilities. Either $s \in S'$ or $s' \notin S'$. There are $|S|$ such choices and all of these choices are independent, so the total number of possibilities is $2^{|S|}$ □

We can still use this principle if the choices aren't quite independent but the number of possibilities for each choice is independent of the previous choices.

Definition 1.10. *For all natural numbers n, we define $n!$ to be $n! = \prod_{j=1}^{n} j$. We define $0! = 1$.*

Definition 1.11. *We define a permutation of a set A to be a one to one function π from A to itself.*

Definition 1.12. *We define S_n (the symmetric group on n elements) to be the set of all permutations $\pi : [n] \to [n]$ of $[n]$.*

Lemma 1.13. *For all $n \in [n]$, $|S_n| = n!$*

Proof. To choose a one to one map $\pi : [n] \to [n]$, we need to choose $\pi(1), \pi(2), \ldots, \pi(n)$. There are n choices for $\pi(1)$. After we choose $\pi(1)$, there are $n - 1$ choices for $\pi(2)$ (as we cannot choose $\pi(2) = \pi(1)$). After we choose $\pi(1)$ and $\pi(2)$, there are $n - 2$ choices for $\pi(3)$ (as we cannot choose $\pi(3) = \pi(1)$ or $\pi(3) = \pi(2)$). Continuing in this way, there are $n - j + 1$ choices for $\pi(j)$. Thus, there are a total of $\prod_{j=1}^{n} n - j + 1 = n * (n - 1) * \ldots * 1 = n!$ possibilities. □

Example 1.14. *We can view a permutation $\pi : [n] \to [n]$ as an ordering $\pi(1), \pi(2), \ldots, \pi(n)$ of the elements in $[n]$. Viewed in this way, the six permutations in S_3 are*

1. $1, 2, 3$
2. $1, 3, 2$
3. $2, 1, 3$
4. $2, 3, 1$
5. $3, 1, 2$
6. $3, 2, 1$

1.1 Partitioning by Cases/Patterns

If all else fails, then we can always break the possibilities into disjoint cases/patterns and count the number of possibilities for each case/pattern.

Example 1.15.
Q: How many ways are there to have 25 *cents with dimes, nickels, and pennies?*

Answer: We can break this up based on the number of dimes we have. For a fixed number of dimes, once we choose the number of nickels, this determines the number of pennies we have, so for a fixed number of dimes the number of possibilities is the number of choices for how many nickels we can have.

1. *If we have* 2 *dimes then we can either have* 0 *or* 1 *nickels. This gives* 2 *possibilities.*
2. *If we have* 1 *dime then we can either have* 0*,* 1*,* 2*, or* 3 *nickels. This gives* 4 *possibilities.*
3. *If we have zero dimes then we can either have* 0*,* 1*,* 2*,* 3*,* 4*, or* 5 *nickels. This gives* 6 *possibilities.*

Adding these possibilities up, the total number of possibilities is $2 + 4 + 6 = 12$*.*

Example 1.16.
Q: How many natural numbers are there between 100 *and* 999 *which have two consecutive digits which are the same?*

Answer: There are three possible patterns for which digits are equal to each other in such a number. These patterns are aab*,* baa*, and* aaa*. For each of these patterns, the number of possibilities is as follows:*

1. *For the first pattern* aab*, there are* 9 *choices for* a *(as we can't choose* $a = 0$*) and then there are* 9 *choices for* b *(as we can't choose* $a = b$*), so we have a total of* 81 *choices.*
2. *For the second pattern* baa*, there are* 9 *choices for* b *(as we can't choose* $b = 0$*) and then there are* 9 *choices for* a *(as we can't choose* $a = b$*), so we have a total of* 81 *choices.*
3. *For the third pattern* aaa*, there are* 9 *choices (as we can't choose* $a = 0$*).*

The total number of natural numbers between 100 *and* 999 *which have two consecutive digits which are the same is* $81 + 81 + 9 = 171$

2 Inclusion/Exclusion

If the cases we are considering aren't quite disjoint, we can use the principle of inclusion/exclusion.

Remark 2.1. *The following arguments are easier to see visually with a Venn diagram.*

Lemma 2.2. *For any sets* A *and* B*,* $|A \cup B| = |A| + |B| - |A \cap B|$

Proof. This can be proved by breaking up $A \cup B$ into three components, $A \setminus B$, $B \setminus A$, and $A \cap B$. We now have the following equations:

1. $|A \cup B| = |A \setminus B| + |B \setminus A| + |A \cap B|$
2. $|A| = |A \setminus B| + |A \cap B|$
3. $|B| = |B \setminus A| + |A \cap B|$

Now observe that $|A| + |B| - |A \cap B| = |A \setminus B| + |B \setminus A| + |A \cap B| = |A \cup B|$. □

Example 2.3. *If a population of students has* 50 *English majors,* 80 *history majors, and* 5 *students who are both English and history majors then the total number of students who are English or history majors is* $50 + 80 - 5 = 125$

Lemma 2.4. *For any sets A, B, and C,* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Proof. This can be proved by breaking up $A \cup B \cup C$ into seven disjoint components:

1. $A \setminus (B \cup C)$
2. $B \setminus (A \cup C)$
3. $C \setminus (A \cup B)$
4. $(A \cap B) \setminus C$
5. $(A \cap C) \setminus B$
6. $(B \cap C) \setminus A$
7. $A \cap B \cap C$

We now have the following equations:

1. $$\begin{aligned} |A \cup B \cup C| = {} & |A \setminus (B \cup C)| + |B \setminus (A \cup C)| + |C \setminus (A \cup B)| \\ & + |(A \cap B) \setminus C| + |(A \cap C) \setminus B| + |(B \cap C) \setminus A| + |A \cap B \cap C| \end{aligned}$$
2. $|A| = |A \setminus (B \cup C)| + |(A \cap B) \setminus C| + |(A \cap C) \setminus B| + |A \cap B \cap C|$
3. $|B| = |B \setminus (A \cup C)| + |(A \cap B) \setminus C| + |(B \cap C) \setminus A| + |A \cap B \cap C|$
4. $|C| = |C \setminus (A \cup B)| + |(A \cap C) \setminus B| + |(B \cap C) \setminus A| + |A \cap B \cap C|$
5. $|A \cap B| = |(A \cap B) \setminus C| + |A \cap B \cap C|$
6. $|A \cap C| = |(A \cap C) \setminus B| + |A \cap B \cap C|$
7. $|B \cap C| = |(B \cap C) \setminus A| + |A \cap B \cap C|$

Let's try and express $|A \cup B \cup C|$ as

$$|A \cup B \cup C| = c_A|A| + c_B|B| + c_C|C| + c_{AB}|A \cap B| + c_{AC}|A \cap C| + c_{BC}|B \cap C| + c_{ABC}|A \cap B \cap C|$$

In order to have the right coefficient of $|A \setminus (B \cup C)|$, $|B \setminus (A \cup C)|$, and $|C \setminus (A \cup B)|$, we must have $c_A = c_B = c_C = 1$. Now observe that

$$\begin{aligned}|A| + |B| + |C| = &|A \setminus (B \cup C)| + |B \setminus (A \cup C)| + |C \setminus (A \cup B)| \\ &+ 2|(A \cap B) \setminus C| + 2|(A \cap C) \setminus B| + 2|(B \cap C) \setminus A| + 3|A \cap B \cap C|\end{aligned}$$

In order to have the right coefficient of $(A \cap B) \setminus C$, $(A \cap C) \setminus B$, and $(B \cap C) \setminus A$, we must have $c_{AB} = c_{AC} = c_{BC} = -1$. Now observe that

$$\begin{aligned}&|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\ &= |A \setminus (B \cup C)| + |B \setminus (A \cup C)| + |C \setminus (A \cup B)| \\ &+ |(A \cap B) \setminus C| + |(A \cap C) \setminus B| + |(B \cap C) \setminus A| + 0 * |A \cap B \cap C|\end{aligned}$$

In order to have the right coefficient of $|A \cap B \cap C|$, we need to add $|A \cap B \cap C|$, so we obtain that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

as needed $\square$

Example 2.5. *If a population of students has the following numbers of English, history, and philosophy majors:*

1. 50 *English majors*
2. 80 *history majors*
3. 60 *philosphy majors*
4. 5 *students who are both English and history majors*
5. 4 *students who are both English and philosophy majors*
6. 7 *students who are both history and philosophy majors*
7. 1 *student who is an English, history, and philosophy major.*

then the total number of students who are English, history, and philosophy majors is $50 + 80 + 60 - 5 - 4 - 7 + 1 = 175$

More generally, the principle of inclusion/exclusion says the following:

Theorem 2.6 (Inclusion/Exclusion). *For all $k \in \mathbb{N}$ and all finite sets $S_1, S_2, \ldots, S_k$,*

$$|\cup_{i \in [k]} S_i| = \sum_{I: I \subseteq [k], I \neq \emptyset} (-1)^{|I|+1} |\cap_{i \in I} S_i|$$