Basic Rules in Combinatorics

Combinatorics: Branch of mathematics which is about counting things.

Rule #1: If we have independent choices, total # of possibilities = Product of (# of possibilities for each choice)

Fundamental example: Recall $A \times B = \{(a,b) : a \in A \wedge b \in B\}$.

$|A \times B| = |A| \cdot |B|$

More examples:

Q: If you have 8 shirts and 6 pairs of pants, how many outfits can you make?

A: $8 \times 6 = 48$

Q: How many 4-digit numbers are there?

A: Digit 1 Digit 2 Digit 3 Digit 4

9 choices $\times$ 10 choices $\times$ 10 choices $\times$ 10 choices $= 9000$

1–9 0–9 0–9 0–9

Alternate solution: Observe that there are $9999 - 1000 + 1 = 9000$ natural numbers between 1000 and 9999.

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Note: For this rule, the choices don't have to be fully independent as long as the number of choices is independent of the previous choices.

Example: How many 3-digit numbers are there whose digits are all distinct?

Answer: Let abc be the three digits.

9 choices for a as $a \in \{1, 2, \dots, 9\}$

9 choices for b as $b \in \{0, 1, \dots, 9\} \setminus \{a\}$

8 choices for c as $c \in \{0, 1, \dots, 9\} \setminus \{a, b\}$

$9 \times 9 \times 8 = 81 \times 8 = 648$

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Rule #2: If we have disjoint sets of choices, total # of choices = sum of (# of choices in each set)

Set theory statement: If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$.

More generally, if $A_1, \ldots, A_n$ are sets such that $\forall i,j \in [n]\ (i \neq j \to A_i \cap A_j = \emptyset)$ then $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$.

Examples:

Q: How many natural numbers between 1 and 25 are either squares or primes?

A: 5 squares: $\{1, 4, 9, 16, 25\}$
9 primes: $\{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

$5 + 9 = 14$

Q: How many 2-digit numbers are there whose digits are either both even or both odd?

A: Both odd: $5 \times 5 = 25$
Both even: $4 \times 5 = 20$

$25 + 20 = 45$