Markov's Inequality

Definition: Given a probability space $(\Omega, \Pr)$ and a random variable X, we say that X is non-negative if $\forall \omega \in \Omega$, $X(\omega) \geq 0$.

Markov's Inequality: Given a non-negative random variable X, $\forall a > 0$, $\Pr(X \geq a) \leq \frac{E[X]}{a}$

Proof: $E[X] = \sum_{\omega \in \Omega} \Pr(\omega) X(\omega)$

$$= \sum_{\omega \in \Omega : X(\omega) < a} \Pr(\omega) X(\omega) + \sum_{\omega \in \Omega : X(\omega) \geq a} \Pr(\omega) X(\omega)$$

$$\geq \sum_{\omega \in \Omega : X(\omega) < a} \Pr(\omega) \cdot 0 + \sum_{\omega \in \Omega : X(\omega) \geq a} \Pr(\omega) \cdot a$$

$$= 0 + a \Pr(X \geq a)$$

$a \Pr(X \geq a) \leq E[X]$ so $\Pr(X \geq a) \leq \frac{E[X]}{a}$

Exercise: Can you give an example where Markov's Inequality is tight, i.e. $\Pr(X \geq a) = \frac{E[X]}{a}$?

Markov's Inequality Example

Example: If we flip a fair coin 6 times and X = # of heads, what bound does Markov's Inequality give on $Pr(X \geq 5)$?

$E[X] = 3$ so taking $a = 5$, by Markov's Inequality $Pr(X \geq 5) \leq \frac{3}{5}$

True probability:

x	0	1	2	3	4	5	6
$Pr(X=x)$	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

$Pr(X \geq 5) = \frac{6}{64} + \frac{1}{64} = \frac{7}{64} \approx 10.9\%$

Note: Applying Markov's Inequality directly to a non-negative random variable X often gives a poor bound, but applying Markov's Inequality to a function of X like $(X - E[X])^2$, X^{2k}, or e^X can be very effective.