Chebyshev's Inequality

Chebyshev's Inequality: For any random variable X and any $a>0$ $Pr(|X-E[X]| \geq a) \leq \frac{Var(X)}{a^2}$

Proof: Apply Markov's Inequality to $Y=(X-E[X])^2$

$E[Y]=E[(X-E[X])^2]=Var(X)$ and $|X-E[X]| \geq a \Leftrightarrow Y \geq a^2$ so

$Pr(|X-E[X]| \geq a) = Pr(Y \geq a^2) \leq \frac{E[Y]}{a^2} = \frac{Var(X)}{a^2}$

Example: If we flip a fair coin 6 times and $X = \#$ of heads, what bound does Chebyshev's Inequality give on $Pr(X \leq 1 \text{ OR } X \geq 5)$?

$Var(X) = \frac{6}{4} = \frac{3}{2}$. By Chebyshev's Inequality, $Pr(|X-E[X]| \geq 2) \leq \frac{Var(X)}{2^2} = \frac{\frac{3}{2}}{4} = \frac{3}{8}$ $= 37.5\%$

True probability: $2P(X \geq 5) = 2 \cdot \frac{7}{64} = \frac{7}{32} \approx 21.9\%$

Exercise: Can you give an example where Chebyshev's Inequality is tight, i.e. $Pr(|X-E[X]| \geq a) = \frac{Var(X)}{a^2}$?